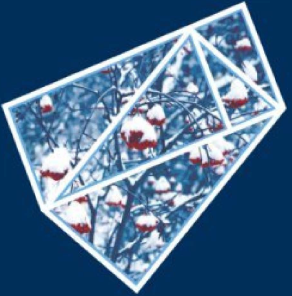
Kiselev's

GEOMETRY

Book I. PLANIMETRY



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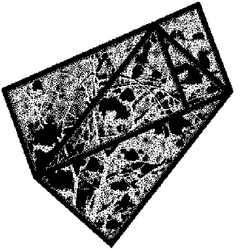

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Translator's Foreword

Those reading these lines are hereby summoned to raise their children to a good command of Elementary Geometry, to be judged by the rigorous standards of the ancient Greek mathematicians.

A magic spell

Mathematics is an ancient culture. It is passed on by each generation to the next. What we now call *Elementary Geometry* was created by Greeks some 2300 years ago and nurtured by them with pride for about a millennium. Then, for another millennium, Arabs were preserving Geometry and transcribing it to the language of *Algebra* that they invented. The effort bore fruit in the Modern Age, when exact sciences emerged through the work of Frenchman Rene Descartes, Englishman Isaac Newton, German Carl Friedrich Gauss, and their contemporaries and followers.

Here is one reason. On the decline of the 19th century, a Scottish professor showed to his class that the mathematical equations, he introduced to explain electricity experiments, admit wave-like solutions. Afterwards a German engineer Heinrich Hertz, who happened to be a student in that class, managed to generate and register the waves. A century later we find that almost every thing we use: GPS, TV, cell-phones, computers, and everything we manufacture, buy, or learn using them, descends from the mathematical discovery made by James Clerk Maxwell.

I gave the above speech at a graduation ceremony at the University of California Berkeley, addressing the class of graduating math majors — and then I cast a *spell* upon them.

Soon there came the realization that without a Magic Wand the spell won't work: I did not manage to find any textbook in English that I could recommend to a young person willing to master Elementary Geometry. This is when the thought of Kiselev's came to mind.

Andrei Petrovich Kiselev (pronounced And-'rei Pet-'ro-vich Ki-se-'lyov) left a unique legacy to mathematics education. Born in 1852 in a provincial Russian town Mzensk, he graduated in 1875 from the Department of Mathematics and Physics of St.-Petersburg University to begin a long career as a math and science teacher and author. His school-level textbooks "A Systematic Course of Arithmetic" [1] [9], "Elementary Algebra" [10], and "Elementary Geometry"(Book I "Planimetry", Book II "Stereometry") [3] were first published in 1884, 1888 and 1892 respectively, and soon gained a leading position in the Russian mathematics education. Revised and published more than a hundred times altogether, the books retained their leadership over many decades both in Tsarist Russia, and after the Revolution of 1917, under the quite different cultural circumstances of the Soviet epoch. A few years prior to Kiselev's death in 1940, his books were officially given the status of *stable*, i.e. main and only textbooks to be used in all schools to teach all teenagers in the totalitarian state with a 200-million population. The books held this status until 1955 (and "Stereometry" even until 1974) when they got replaced in this capacity by less successful clones written by more Soviet authors. Yet "Planimetry" remained the favorite under-the-desk choice of many teachers and a must for honors geometry students. In the last decade, Kiselev's "Geometry," which has long become a rarity, was reprinted by several major publishing houses in Moscow and St.-Petersburg in both versions: for teachers [6, 8] as an authentic pedagogical heritage, and for students [5, 7] as a textbook tailored to fit the currently active school curricula. In the post-Soviet educational market, Kiselev's "Geometry" continues to compete successfully with its own grandchildren.

What is the secret of such ageless vigor? There are several.

Kiselev himself formulated the following three key virtues of good textbooks: *precision*, *simplicity*, *conciseness*. And *competence in the subject* — for we must now add this fourth criterion, which could have been taken for granted a century ago.

Acquaintance with programs and principles of math education being developed by European mathematicians was another of Kiselev's assets. In his preface to the first edition of "Elementary Geometry," in addition to domestic and translated textbooks, Kiselev quotes ten geometry courses in French and German published in the previous decade.

Yet another vital elixir that prolongs the life of Kiselev's work was the continuous effort of the author himself and of the editors of later reprints to improve and update the books, and to accommodate the teachers' requests, curriculum fluctuations and pressures of the 20th century classroom.

Last but not least, deep and beautiful geometry is the most efficient preservative. Compared to the first textbook in this subject: the "Elements" [1], which was written by *Euclid of Alexandria* in the 3rd century B.C., and whose spirit and structure are so faithfully represented in Kiselev's "Geometry," the latter is quite young.

Elementary geometry occupies a singular place in secondary education. The acquiring of superb reasoning skills is one of those benefits from study-

[1] The numbers in brackets refer to the bibliography on p. 235.

ing geometry whose role reaches far beyond mathematics education *per se*. Another one is the unlimited opportunity for nurturing creative thinking (thanks to the astonishingly broad difficulty range of elementary geometry problems that have been accumulated over the decades). Fine learning habits of those who dared to face the challenge remain always at work for them. A lack thereof in those who missed it becomes hard to compensate by studying anything else. Above all, elementary geometry conveys the essence and power of the *theoretical method* in its purest, yet intuitively transparent and aesthetically appealing, form. Such high expectations seem to depend however on the appropriate framework: a textbook, a teacher, a culture.

In Russia, the adequate framework emerged apparently in the mid-thirties, with Kiselev's books as the key component. After the 2nd World War, countries of Eastern Europe and the Peoples Republic of China, adapted to their classrooms math textbooks based on Soviet programs. Thus, one way or another, Kiselev's "Geometry" has served several generations of students and teachers in a substantial portion of the planet. It is the time to make the book available to the English reader.

"Planimetry," targeting the age group of current 7–9th-graders, provides a concise yet crystal-clear presentation of elementary plane geometry, in all its aspects which usually appear in modern high-school geometry programs. The reader's mathematical maturity is gently advanced by commentaries on the nature of mathematical reasoning distributed wisely throughout the book. Student's competence is reinforced by generously supplied exercises of varying degree of challenge. Among them, *straightedge and compass* constructions play a prominent role, because, according to the author, they are essential for animating the subject and cultivating students' taste. The book is marked with the general sense of measure (in both selections and omissions), and non-cryptic, unambiguous language. This makes it equally suitable for independent study, teachers' professional development, or a regular school classroom. The book was indeed designed and tuned to be *stable*.

Hopefully the present adaptation retains the virtues of the original. I tried to follow it pretty closely, alternating between several available versions [3, 4, 5, 7, 8] when they disagreed. Yet authenticity of translation was not the goal, and I felt free to deviate from the source when the need occurred.

The most notable change is the significant extension and rearrangement of exercise sections to comply with the US tradition of making textbook editions self-contained (in Russia separate problem books are in fashion).

Also, I added or redesigned a few sections to represent material which found its way to geometry curricula rather recently.

Finally, having removed descriptions of several obsolete drafting devices (such as a pantograph), I would like to share with the reader the following observation.

In that remote, Kiselevian past, when Elementary Geometry was the most reliable ally of every engineer, the straightedge and compass were the main items in his or her drafting toolbox. The craft of blueprint drafting has long gone thanks to the advance of computers. Consequently, all 267 diagrams in the present edition are produced with the aid of graphing software *Xfig*. Still, Elementary Geometry is manifested in their design in multiple ways. Obviously, it is inherent in all modern technologies through the "custody chain": Euclid – Descartes – Newton – Maxwell. Plausibly, it awakened the innovative powers of the many scientists and engineers who invented and created computers. Possibly, it was among the skills of the authors of *Xfig*. Yet, symbolically enough, the most reliable way of drawing a diagram on the computer screen is to use electronic surrogates of the straightedge and compass and follow literally the prescriptions given in the present book, often in the very same theorem that the diagram illustrates. This brings us back to Euclid of Alexandria, who was the first to describe the theorem, and to the task of passing on *his* culture.

I believe that the book you are holding in your hands gives everyone a fair chance to share in the "custody." This is my Magic Wand, and now I can cast my spell.

Alexander Givental
Department of Mathematics
University of California Berkeley
April, 2006

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Hippocrates of Chios	*470 – 410 B.C.*
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Ferdinand Lindemann	*1852 – 1939*
Heinrich Hertz	*1857 – 1894*

Introduction

1. Geometric figures. The part of space occupied by a physical object is called a **geometric solid.**

A geometric solid is separated from the surrounding space by a **surface.**

A part of the surface is separated from an adjacent part by a **line.**

A part of the line is separated from an adjacent part by a **point.**

The geometric solid, surface, line and point do not exist separately. However by way of abstraction we can consider a surface independently of the geometric solid, a line — independently of the surface, and the point — independently of the line. In doing so we should think of a surface as having no thickness, a line — as having neither thickness nor width, and a point — as having no length, no width, and no thickness.

A set of points, lines, surfaces, or solids positioned in a certain way in space is generally called a **geometric figure.** Geometric figures can move through space without change. Two geometric figures are called **congruent,** if by moving one of the figures it is possible to superimpose it onto the other so that the two figures become identified with each other in all their parts.

2. Geometry. A theory studying properties of geometric figures is called **geometry,** which translates from Greek as *land-measuring.* This name was given to the theory because the main purpose of geometry in antiquity was to measure distances and areas on the Earth's surface.

First concepts of geometry as well as their basic properties, are introduced as idealizations of the corresponding common notions and everyday experiences.

3. The plane. The most familiar of all surfaces is the flat surface, or the **plane.** The idea of the plane is conveyed by a window pane, or the water surface in a quiet pond.

We note the following property of the plane: *One can superimpose a plane on itself or any other plane in a way that takes one given point to any other given point, and this can also be done after flipping the plane upside down.*

4. The straight line. The most simple line is the **straight line**. The image of a thin thread stretched tight or a ray of light emitted through a small hole give an idea of what a straight line is. The following fundamental property of the straight line agrees well with these images:

For every two points in space, there is a straight line passing through them, and such a line is unique.

It follows from this property that:

If two straight lines are aligned with each other in such a way that two points of one line coincide with two points of the other, then the lines coincide in all their other points as well (because otherwise we would have two distinct straight lines passing through the same two points, which is impossible).

For the same reason, *two straight lines can intersect at most at one point.*

A straight line can lie in a plane. The following holds true:

If a straight line passes through two points of a plane, then all points of this line lie in this plane.

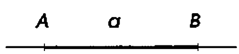

Figure 1

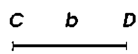

Figure 2

Figure 3

5. The unbounded straight line. Ray. Segment. Thinking of a straight line as extended indefinitely in both directions, one calls it an **infinite** (or **unbounded**) straight line.

A straight line is usually denoted by two uppercase letters marking any two points on it. One says "the line AB" or "BA" (Figure 1).

A part of the straight line bounded on both sides is called a **straight segment**. It is usually denoted by two letters marking its endpoints (the segment CD, Figure 2). Sometimes a straight line or a segment is denoted by one (lowercase) letter; one may say "the straight line a, the segment b."

Usually instead of "unbounded straight line" and "straight segment" we will simply say **line** and **segment** respectively.

Sometimes a straight line is considered which terminates in one direction only, for instance at the endpoint E (Figure 3). Such a straight line is called a **ray** (or **half-line**) drawn from E.

6. Congruent and non-congruent segments. *Two segments are congruent if they can be laid one onto the other so that their endpoints coincide.* Suppose for example that we put the segment AB onto the segment CD (Figure 4) by placing the point A at the point C and aligning the ray AB with the ray CD. If, as a result of this, the points B and D merge, then the segments AB and CD are congruent. Otherwise they are not congruent, and the one which makes a part of the other is considered smaller.

Figure 4

To mark on a line a segment congruent to a given segment, one uses the **compass**, a drafting device which we assume familiar to the reader.

7. Sum of segments. The sum of several given segments (AB, CD, EF, Figure 5) is a segment which is obtained as follows. On a line, pick any point M and starting from it mark a segment MN congruent to AB, then mark the segments NP congruent to CD, and PQ congruent to EF, both going in the same direction as MN. Then the segment MQ will be the sum of the segments AB, CD and EF (which are called **summands** of this sum). One can similarly obtain the sum of any number of segments.

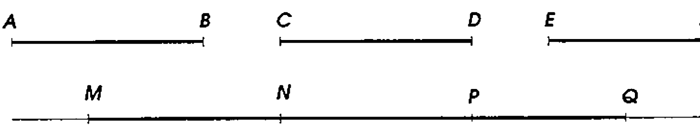

Figure 5

The sum of segments has the same properties as the sum of numbers. In particular it does not depend on the order of the summands (the **commutativity** law) and remains unchanged when some of the summands are replaced with their sum (the **associativity** law). For instance:

$$AB + CD + EF = AB + EF + CD = EF + CD + AB = \ldots$$

and

$$AB + CD + EF = AB + (CD + EF) = CD + (AB + EF) = \ldots.$$

8. Operations with segments. The concept of addition of segments gives rise to the concept of subtraction of segments, and multiplication and division of segments by a whole number. For example, the difference of AB and CD (if $AB > CD$) is a segment whose sum with CD is congruent to AB; the product of the segment AB with the number 3 is the sum of three segments each congruent to AB; the quotient of the segment AB by the number 3 is a third part of AB.

If given segments are measured by certain linear units (for instance, centimeters), and their lengths are expressed by the corresponding numbers, then the length of the sum of the segments is expressed by the sum of the numbers measuring these segments, the length of the difference is expressed by the difference of the numbers, etc.

9. The circle. If, setting the compass to an arbitrary step and, placing its pin leg at some point O of the plane (Figure 6), we begin to turn the compass around this point, then the other leg equipped with a pencil touching the plane will describe on the plane a continuous curved line all of whose points are the same distance away from O. This curved line is called a **circle**, and the point O — its **center**. A segment (OA, OB, OC in Figure 6) connecting the center with a point of the circle is called a **radius**. All radii of the same circle are congruent to each other.

Circles described by the compass set to the same radius are congruent because by placing their centers at the same point one will identify such circles with each other at all their points.

A line (MN, Figure 6) intersecting the circle at any two points is called a **secant**.

A segment (EF) both of whose endpoints lie on the circle is called a **chord**.

A chord (AD) passing through the center is called a **diameter**. A diameter is the sum of two radii, and therefore all diameters of the same circle are congruent to each other.

A part of a circle contained between any two points (for example, EmF) is called an **arc**.

The chord connecting the endpoints of an arc is said to **subtend** this arc.

An arc is sometimes denoted by the sign $\frown$; for instance, one writes: $\overset{\frown}{EmF}$.

The part of the plane bounded by a circle is called a **disk.**[2]

The part of a disk contained between two radii (the shaded part COB in Figure 6) is called a **sector**, and the part of the disk cut off by a secant (the part EmF) is called a **disk segment**.

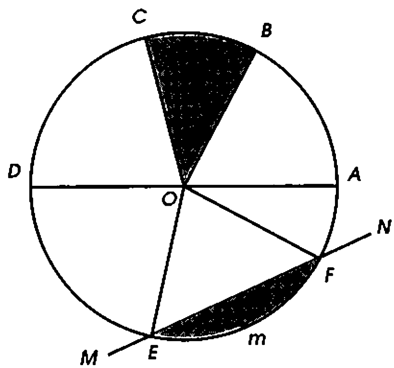

Figure 6

10. Congruent and non-congruent arcs. *Two arcs of the same circle (or of two congruent circles) are congruent if they can be aligned so that their endpoints coincide.* Indeed, suppose that we align the arc AB (Figure 7) with the arc CD by identifying the point A with the point C and directing the arc AB along the arc CD. If, as a result of this, the endpoints B and D coincide, then all the intermediate points of these arcs will coincide as well, since they are the same distance away from the center, and therefore $\overset{\frown}{AB}=\overset{\frown}{CD}$. But if B and D do not coincide, then the arcs are not congruent, and the one which is a part of the other is considered smaller.

11. Sum of arcs. The sum of several given arcs of the same radius is defined as an arc of that same radius which is composed from parts congruent respectively to the given arcs. Thus, pick an arbitrary point M (Figure 7) of the circle and mark the part MN

[2]Often the word "circle" is used instead of "disk." However one should avoid doing this since the use of the same term for different concepts may lead to mistakes.

congruent to AB. Next, moving in the same direction along the circle, mark the part NP congruent to CD. Then the arc MP will be the sum of the arcs AB and CD.

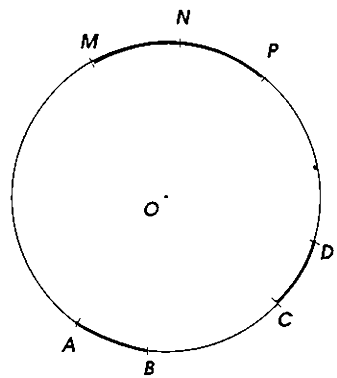

Figure 7

Adding arcs of the same radius one may encounter the situation when the sum of the arcs does not fit in the circle and one of the arcs partially covers another. In this case the sum will be an arc greater than the whole circle. For example, adding the arcs AmB and CnD (Figure 8) we obtain the arc consisting of the whole circle and the arc AD.

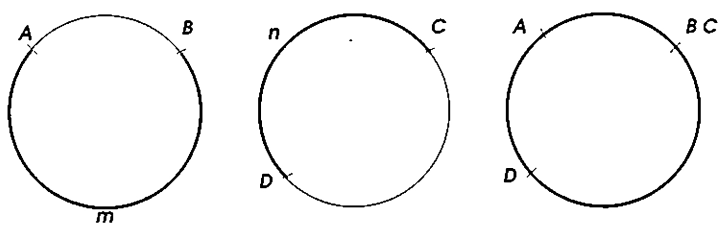

Figure 8

Similarly to addition of line segments, addition of arcs obeys the commutativity and associativity laws.

From the concept of addition of arcs one derives the concepts of subtraction of arcs, and multiplication and division of arcs by a whole number the same way as it was done for line segments.

12. Divisions of geometry. The subject of geometry can be divided into two parts: **plane geometry**, or **planimetry**, and **solid geometry**, or **stereometry**. Planimetry studies properties of those geometric figures all of whose elements fit the same plane.

EXERCISES

1. Give examples of geometric solids bounded by one, two, three, four planes (or parts of planes).

2. Show that if a geometric figure is congruent to another geometric figure, which is in its turn congruent to a third geometric figure, then the first geometric figure is congruent to the third.

3. Explain *why* two straight lines in space can intersect at most at one point.

4. Referring to §4, show that a plane not containing a given straight line can intersect it at most at one point.

*5.** [3] Give an example of a surface other than the plane which, like the plane, can be superimposed on itself in a way that takes any one given point to any other given point.
Remark: The required example is not unique.

6. Referring to §4, show that for any two points of a plane, there is a straight line lying *in this plane* and passing through them, and that such a line is unique.

7. Use a straightedge to draw a line passing through two points given on a sheet of paper. Figure out how to check that the line is really straight.
Hint: Flip the straightedge upside down.

*8.** Fold a sheet of paper and, using the previous problem, check that the edge is straight. Can you explain why the edge of a folded paper is straight?
Remark: There may exist several correct answers to this question.

9. Show that for each point lying in a plane there is a straight line lying in this plane and passing through this point. How many such lines are there?

10. Find surfaces other than the plane which, like the plane, together with each point lying on the surface contain a straight line passing through this point.
Hint: One can obtain such surfaces by bending a sheet of paper.

11. Referring to the definition of congruent figures given in §1, show that any two infinite straight lines are congruent; that any two rays are congruent.

12. On a given line, mark a segment congruent to four times a given segment, using a compass as few times as possible.

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[3] Stars * mark those exercises which we consider more difficult.

13. Is the sum (difference) of given segments unique? Give an example of two distinct segments which both are sums of the given segments. Show that these distinct segments are congruent.

14. Give an example of two non-congruent arcs whose endpoints coincide. Can such arcs belong to non-congruent circles? to congruent circles? to the same circle?

15. Give examples of non-congruent arcs subtended by congruent chords. Are there non-congruent chords subtending congruent arcs?

16. Describe explicitly the operations of subtraction of arcs, and multiplication and division of an arc by a whole number.

17. Follow the descriptions of operations with arcs, and show that multiplying a given arc by 3 and then dividing the result by 2, we obtain an arc congruent to the arc resulting from the same operations performed on the given arc in the reverse order.

18. Can sums (differences) of respectively congruent line segments, or arcs, be non-congruent? Can sums (differences) of respectively non-congruent segments, or arcs be congruent?

19. Following the definition of sum of segments or arcs, explain why addition of segments (or arcs) obeys the commutativity law.
Hint: Identify a segment (or arc) AB with BA.

Chapter 1

THE STRAIGHT LINE

1 Angles

13. Preliminary concepts. A figure formed by two rays drawn from the same point is called an **angle**. The rays which form the angle are called its **sides**, and their common endpoint is called the **vertex** of the angle. One should think of the sides as extending away from the vertex indefinitely.

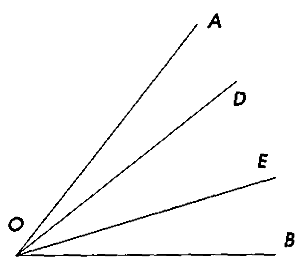

Figure 9

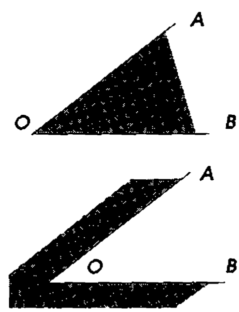

Figure 10

An angle is usually denoted by three uppercase letters of which the middle one marks the vertex, and the other two label a point on each of the sides. One says, e.g.: "the angle AOB" or "the angle BOA" (Figure 9). It is possible to denote an angle by one letter marking the vertex provided that no other angles with the same vertex are present on the diagram. Sometimes we will also denote an angle by a number placed inside the angle next to its vertex.

The sides of an angle divide the whole plane containing the angle into two regions. One of them is called the **interior** region of the angle, and the other is called the **exterior** one. Usually the interior region is considered the one that contains the segments joining any two points on the sides of the angle, e.g. the points A and B on the sides of the angle AOB (Figure 9). Sometimes however one needs to consider the other part of the plane as the interior one. In such cases a special comment will be made regarding which region of the plane is considered interior. Both cases are represented separately in Figure 10, where the interior region in each case is shaded.

Rays drawn from the vertex of an angle and lying in its interior (OD, OE, Figure 9) form new angles (AOD, DOE, EOB) which are considered to be parts of the angle (AOB).

In writing, the word "angle" is often replaced with the symbol $\angle$. For instance, instead of "angle AOB" one may write: $\angle AOB$.

14. Congruent and non-congruent angles. In accordance with the general definition of congruent figures (§1) *two angles are considered congruent if by moving one of them it is possible to identify it with the other.*

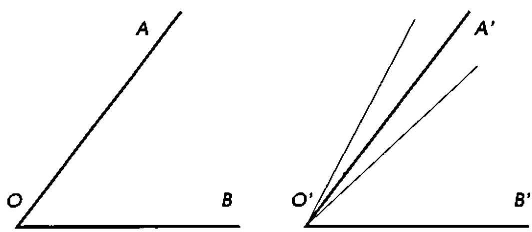

Figure 11

Suppose, for example, that we lay the angle AOB onto the angle $A'O'B'$ (Figure 11) in a way such that the vertex O coincides with O', the side OB goes along OB', and the interior regions of both angles lie on the same side of the line $O'B'$. If OA turns out to coincide with $O'A'$, then the angles are congruent. If OA turns out to lie inside or outside the angle $A'O'B'$, then the angles are non-congruent, and the one, that lies inside the other is said to be **smaller**.

15. Sum of angles. The sum of angles AOB and $A'O'B'$ (Figure 12) is an angle defined as follows. Construct an angle MNP congruent to the given angle AOB, and attach to it the angle PNQ, congruent to the given angle $A'O'B'$, as shown. Namely, the angle

MNP should have with the angle PNQ the same vertex N, a common side NP, and the interior regions of both angles should lie on the opposite sides of the common ray NP. Then the angle MNQ is called the sum of the angles AOB and $A'O'B'$. The interior region of the sum is considered the part of the plane comprised by the interior regions of the summands. This region contains the common side (NP) of the summands. One can similarly form the sum of three and more angles.

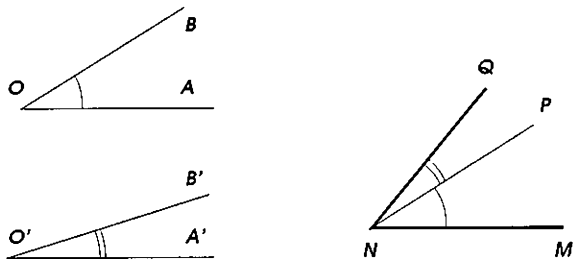

Figure 12

Addition of angles obeys the commutativity and associativity laws just the same way addition of segments does. From the concept of addition of angles one derives the concept of subtraction of angles, and multiplication and division of angles by a whole number.

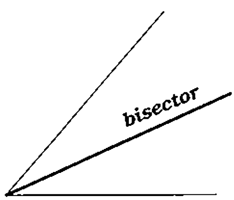

Figure 13

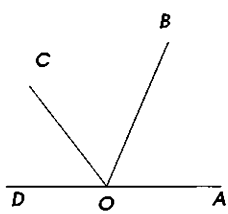

Figure 14

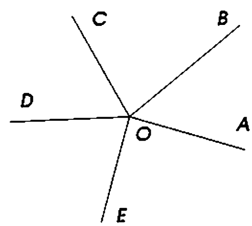

Figure 15

Very often one has to deal with the ray which divides a given angle into halves; this ray is called the **bisector** of the angle (Figure 13).

16. Extension of the concept of angle. When one computes the sum of angles some cases may occur which require special attention.

(1) It is possible that after addition of several angles, say, the

three angles: AOB, BOC and COD (Figure 14), the side OD of the angle COD will happen to be the continuation of the side OA of the angle AOB. We will obtain therefore the figure formed by two half-lines (OA and OD) drawn from the same point (O) and continuing each other. Such a figure is also considered an angle and is called a **straight angle**.

(2) It is possible that after the addition of several angles, say, the five angles: AOB, BOC, COD, DOE and EOA (Figure 15) the side OA of the angle EOA will happen to coincide with the side OA of the angle AOB. The figure formed by such rays (together with the whole plane surrounding the vertex O) is also considered an angle and is called a **full angle**.

(3) Finally, it is possible that added angles will not only fill in the whole plane around the common vertex, but will even overlap with each other, covering the plane around the common vertex for the second time, for the third time, and so on. Such an angle sum is congruent to one full angle added with another angle, or congruent to two full angles added with another angle, and so on.

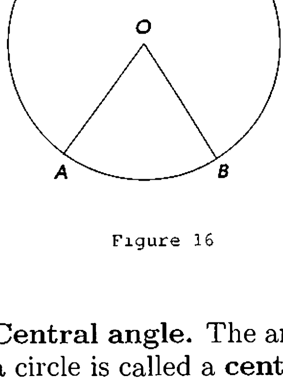

Figure 16

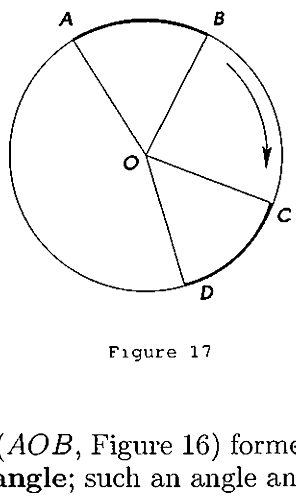

Figure 17

17. Central angle. The angle (AOB, Figure 16) formed by two radii of a circle is called a **central angle**; such an angle and the arc contained between the sides of this angle are said to *correspond* to each other.

Central angles and their corresponding arcs have the following properties.

In one circle, or two congruent circles:

(1) *If central angles are congruent, then the corresponding arcs are congruent;*

(2) *Vice versa, if the arcs are congruent, then the corre-*

sponding central angles are congruent.

Let $\angle AOB = \angle COD$ (Figure 17); we need to show that the arcs AB and CD are congruent too. Imagine that the sector AOB is rotated about the center O in the direction shown by the arrow until the radius OA coincides with OC. Then due to the congruence of the angles, the radius OB will coincide with OD; therefore the arcs AB and CD will coincide too, i.e. they are congruent.

The second property is established similarly.

18. Circular and angular degrees. Imagine that a circle is divided into 360 congruent parts and all the division points are connected with the center by radii. Then around the center, 360 central angles are formed which are congruent to each other as central angles corresponding to congruent arcs. Each of these arcs is called a **circular degree**, and each of those central angles is called an **angular degree**. Thus one can say that a circular degree is 1/360th part of the circle, and the angular degree is the central angle corresponding to it.

The degrees (both circular and angular) are further subdivided into 60 congruent parts called **minutes**, and the minutes are further subdivided into 60 congruent parts called **seconds**.

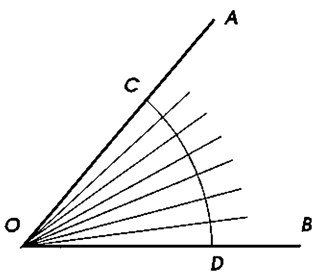

Figure 18

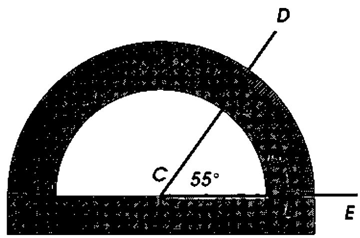

Figure 19

19. Correspondence between central angles and arcs. Let AOB be some angle (Figure 18). Between its sides, draw an arc CD of arbitrary radius with the center at the vertex O. Then the angle AOB will become the central angle corresponding to the arc CD. Suppose, for example, that this arc consists of 7 circular degrees (shown enlarged in Figure 18). Then the radii connecting the division points with the center obviously divide the angle AOB into 7 angular degrees. More generally, one can say that *an angle is measured by the arc corresponding to it*, meaning that an angle contains as many angular degrees, minutes and seconds as the corresponding arc contains circular degrees, minutes and seconds. For instance, if the arc CD contains 20 degrees 10 minutes and 15 seconds of circular units, then the angle AOB consists of 20 degrees 10 minutes and 15 seconds of angular units, which is customary to express as: $\angle AOB = 20°10'15''$, using the symbols °, ′ and ″ to denote degrees, minutes and seconds respectively.

Units of angular degree do not depend on the radius of the circle. Indeed, adding 360 angular degrees following the summation rule described in §15, we obtain the full angle at the center of the circle. Whatever the radius of the circle, this full angle will be the same. Thus one can say that an angular degree is 1/360th part of the full angle.

20. Protractor. This device (Figure 19) is used for measuring angles. It consists of a semi-disk whose arc is divided into 180°. To measure the angle DCE, one places the protractor onto the angle in a way such that the center of the semi-disk coincides with the vertex of the angle, and the radius CB lies on the side CE. Then the number of degrees in the arc contained between the sides of the angle DCE shows the measure of the angle. Using the protractor one can also draw an angle containing a given number of degrees (e.g. the angle of 90°, 45°, 30°, etc.).

EXERCISES

20. Draw any angle and, using a protractor and a straightedge, draw its bisector.

21. In the exterior of a given angle, draw another angle congruent to it. Can you do this in the interior of the given angle?

22. How many common sides can two distinct angles have?

23. Can two non-congruent angles contain 55 angular degrees each?

24. Can two non-congruent arcs contain 55 circular degrees each? What if these arcs have the same radius?

25. Two straight lines intersect at an angle containing 25°. Find the measures of the remaining three angles formed by these lines.

26. Three lines passing through the same point divide the plane into six angles. Two of them turned out to contain 25° and 55° respectively. Find the measures of the remaining four angles.

*27.** Using only compass, construct a 1° arc on a circle, if a 19° arc of this circle is given.

2 Perpendicular lines

21. Right, acute and obtuse angles. An angle of 90° (congruent therefore to one half of the straight angle or to one quarter of the full angle) is called a **right angle**. An angle smaller than the right one is called **acute**, and a greater than right but smaller than straight is called **obtuse** (Figure 20).

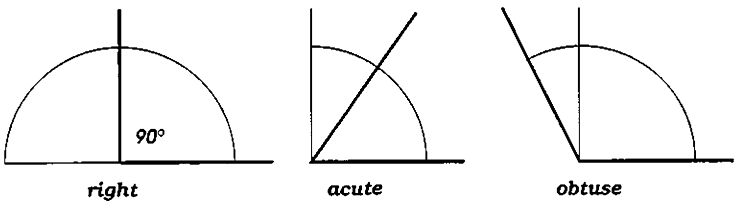

Figure 20

All right angles are, of course, *congruent to each other* since they contain the same number of degrees.

The measure of a right angle is sometimes denoted by d (the initial letter of the French word *droit* meaning "right").

22. Supplementary angles. Two angles (AOB and BOC, Figure 21) are called **supplementary** if they have one common side, and their remaining two sides form continuations of each other. Since the sum of such angles is a straight angle, *the sum of two supplementary angles is* 180° (in other words it is congruent to the sum of two right angles).

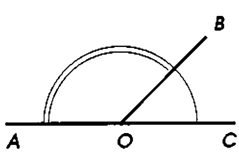

Figure 21

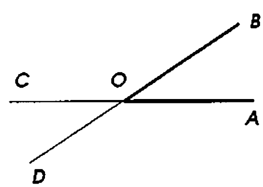

Figure 22

For each angle one can construct two supplementary angles. For example, for the angle AOB (Figure 22), prolonging the side AO we obtain one supplementary angle BOC, and prolonging the side BO we obtain another supplementary angle AOD. *Two angles supplementary to the same one are congruent to each other,* since they both contain the same number of degrees, namely the number that supplements the number of degrees in the angle AOB to 180° contained in a straight angle.

If AOB is a right angle (Figure 23), i.e. if it contains 90°, then each of its supplementary angles COB and AOD must also be right, since it contains $180° - 90°$, i.e. 90°. The fourth angle COD has to be right as well, since the three angles AOB, BOC and AOD contain 270° altogether, and therefore what is left from 360° for the fourth angle COD is 90° too. Thus, *if one of the four angles formed by two intersecting lines* (AC and BD, Figure 23) *is right, then the other three angles must be right as well.*

23. A perpendicular and a slant. In the case when two supplementary angles are not congruent to each other, their common side (OB, Figure 24) is called a **slant** [1] to the line (AC) containing the other two sides. When, however, the supplementary angles are congruent (Figure 25) and when, therefore, each of the angles is right, the common side is called a **perpendicular** to the line containing the other two sides. The common vertex (O) is called the **foot of the slant** in the first case, and the **foot of the perpendicular** in the second.

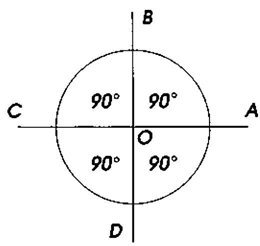

Figure 23

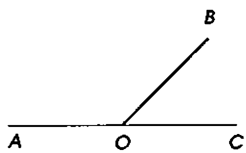

Figure 24

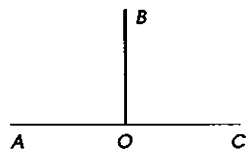

Figure 25

Two lines (AC and BD, Figure 23) intersecting at a right angle are called **perpendicular** to each other. The fact that the line AC is perpendicular to the line BD is written: $AC \perp BD$.

Remarks. (1) If a perpendicular to a line AC (Figure 25) needs to be drawn through a point O lying on this line, then the perpendicular is said to be "erected" to the line AC, and if the perpendicular needs to be drawn through a point B lying outside the line, then the perpendicular is said to be "dropped" to the line (no matter if it is upward, downward or sideways).

[1] Another name used for a slant is an **oblique line**.

(2) Obviously, at any given point of a given line, on either side of it, one can erect a perpendicular, and such a perpendicular is unique.

24. Let us prove that ***from any point lying outside a given line one can drop a perpendicular to this line, and such perpendicular is unique.***

Let a line AB (Figure 26) and an arbitrary point M outside the line be given. We need to show that, first, one can drop a perpendicular from this point to AB, and second, that there is only one such perpendicular.

Imagine that the diagram is folded so that the upper part of it is identified with the lower part. Then the point M will take some position N. Mark this position, unfold the diagram to the initial form and then connect the points M and N by a line. Let us show now that the resulting line MN is perpendicular to AB, and that any other line passing through M, for example MD, is not perpendicular to AB. For this, fold the diagram again. Then the point M will merge with N again, and the points C and D will remain in their places. Therefore the line MC will be identified with NC, and MD with ND. It follows that $\angle MCB = \angle BCN$ and $\angle MDC = \angle CDN$.

But the angles MCB and BCN are supplementary. Therefore each of them is right, and hence $MN \perp AB$. Since MDN is not a straight line (because there can be no two straight lines connecting the points M and N), then the sum of the two congruent angles MDC and CDN is not equal to $2d$. Therefore the angle MDC is not right, and hence MD is not perpendicular to AB. Thus one can drop no other perpendicular from the point M to the line AB.

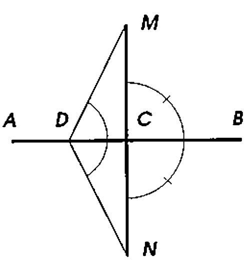

Figure 26

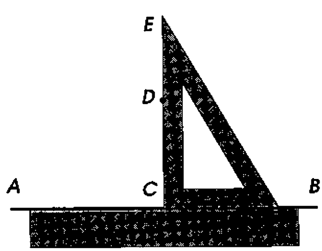

Figure 27

25. The drafting triangle. For practical construction of a perpendicular to a given line it is convenient to use a **drafting triangle** made to have one of its angles right. To draw the perpendicular to a line AB (Figure 27) through a point C lying on this line, or through a point D taken outside of this line, one can align a straightedge with the line AB, the drafting triangle with the straightedge, and then slide the triangle along the straightedge until the other side of the right angle hits the point C or D, and then draw the line CE.

26. Vertical angles. Two angles are called **vertical** if the sides of one of them form continuations of the sides of the other. For instance, at the intersection of two lines AB and CD (Figure 28) two pairs of vertical angles are formed: AOD and COB, AOC and DOB (and four pairs of supplementary angles).

Two vertical angles are congruent to each other (for example, $\angle AOD = \angle BOC$) since each of them is supplementary to the same angle (to $\angle DOB$ or to $\angle AOC$), and such angles, as we have seen (§22), are congruent to each other.

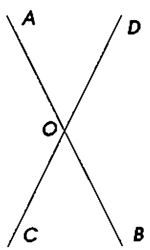

Figure 28

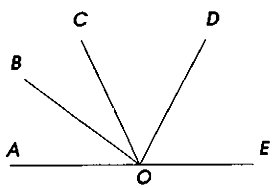

Figure 29

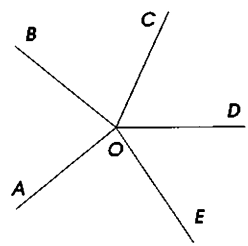

Figure 30

27. Angles that have a common vertex. It is useful to remember the following simple facts about angles that have a common vertex:

(1) *If the sum of several angles* (AOB, BOC, COD, DOE, Figure 29) *that have a common vertex is congruent to a straight angle, then the sum is* $2d$, *i.e.* $180°$.

(2) *If the sum of several angles* (AOB, BOC, COD, DOE, EOA, Figure 30) *that have a common vertex is congruent to the full angle, then it is* $4d$, *i.e.* $360°$.

(3) *If two angles* (AOB and BOC, Figure 24) *have a common vertex* (O) *and a common side* (OB) *and add up to* $2d$ (i.e. $180°$), *then their two other sides* (AO and OC) *form continuations of each other* (i.e. such angles are supplementary).

EXERCISES

28. Is the sum of the angles $14°24'44''$ and $75°35'25''$ acute or obtuse?

29. Five rays drawn from the same point divide the full angle into five congruent parts. How many different angles do these five rays form? Which of these angles are congruent to each other? Which of them are acute? Obtuse? Find the degree measure of each of them.

30. Can both angles, whose sum is the straight angle, be acute? obtuse?

31. Find the smallest number of acute (or obtuse) angles which add up to the full angle.

32. An angle measures $38°20'$; find the measure of its supplementary angles.

33. One of the angles formed by two intersecting lines is $2d/5$. Find the measures of the other three.

34. Find the measure of an angle which is congruent to twice its supplementary one.

35. Two angles ABC and CBD having the common vertex B and the common side BC are positioned in such a way that they do not cover one another. The angle $ABC = 100°20'$, and the angle $CBD = 79°40'$. Do the sides AB and BD form a straight line or a bent one?

36. Two distinct rays, perpendicular to a given line, are erected at a given point. Find the measure of the angle between these rays.

37. In the interior of an obtuse angle, two perpendiculars to its sides are erected at the vertex. Find the measure of the obtuse angle, if the angle between the perpendiculars is $4d/5$.

Prove:

38. Bisectors of two supplementary angles are perpendicular to each other.

39. Bisectors of two vertical angles are continuations of each other.

40. If at a point O of the line AB (Figure 28) two congruent angles AOD and BOC are built on the opposites sides of AB, then their sides OD and OC form a straight line.

41. If from the point O (Figure 28) rays OA, OB, OC and OD are constructed in such a way that $\angle AOC = \angle DOB$ and $\angle AOD = \angle COB$, then OB is the continuation of OA, and OD is the continuation of OC.
Hint: Apply §27, statements 2 and 3.

3 Mathematical propositions

28. Theorems, axioms, definitions. From what we have said so far one can conclude that some geometric statements we consider quite obvious (for example, the properties of planes and lines in §3 and §4) while some others are established by way of reasoning (for example, the properties of supplementary angles in §22 and vertical angles in §26). In geometry, this process of reasoning is a principal way to discover properties of geometric figures. It would be instructive therefore to acquaint yourself with the forms of reasoning usual in geometry.

All facts established in geometry are expressed in the form of propositions. These propositions are divided into the following types.

Definitions. Definitions are propositions which explain what meaning one attributes to a name or expression. For instance, we have already encountered the definitions of central angle, right angle, perpendicular lines, etc.

Axioms. Axioms [2] are those facts which are accepted without proof. This includes, for example, some propositions we encountered earlier (§4): through any two points there is a unique line; if two points of a line lie in a given plane then all points of this line lie in the same plane.

Let us also mention the following axioms which apply to any kind of quantities:

if each of two quantities is equal to a third quantity, then these two quantities are equal to each other;

if the same quantity is added to or subtracted from equal quantities, then the equality remains true;

if the same quantity is added to or subtracted from unequal quantities, then the inequality remains unchanged, i.e. the greater quantity remains greater.

Theorems. Theorems are those propositions whose truth is found only through a certain reasoning process (proof). The following propositions may serve as examples:

if in one circle or two congruent circles some central angles are congruent, then the corresponding arcs are congruent;

if one of the four angles formed by two intersecting lines turns out to be right, then the remaining three angles are right as well.

[2] In geometry, some axioms are traditionally called **postulates**.

Corollaries. Corollaries are those propositions which follow directly from an axiom or a theorem. For instance, it follows from the axiom "there is only one line passing through two points" that "two lines can intersect at one point at most."

29. The content of a theorem. In any theorem one can distinguish two parts: the hypothesis and the conclusion. The **hypothesis** expresses what is considered given, the **conclusion** what is required to prove. For example, in the theorem "if central angles are congruent, then the corresponding arcs are congruent" the hypothesis is the first part of the theorem: "if central angles are congruent," and the conclusion is the second part: "then the corresponding arcs are congruent;" in other words, it is given (known to us) that the central angles are congruent, and it is required to prove that under this hypothesis the corresponding arcs are congruent.

The hypothesis and the conclusion of a theorem may sometimes consist of several separate hypotheses and conclusions; for instance, in the theorem "if a number is divisible by 2 and by 3, then it is divisible by 6," the hypothesis consists of two parts: "if a number is divisible by 2" and "if the number is divisible by 3."

It is useful to notice that any theorem can be rephrased in such a way that the hypothesis will begin with the word "if," and the conclusion with the word "then." For example, the theorem "vertical angles are congruent" can be rephrased this way: "*if* two angles are vertical, *then* they are congruent."

30. The converse theorem. The theorem converse to a given theorem is obtained by replacing the hypothesis of the given theorem with the conclusion (or some part of the conclusion), and the conclusion with the hypothesis (or some part of the hypothesis) of the given theorem. For instance, the following two theorems are converse to each other:

If central angles are congruent, then the corresponding arcs are congruent.

If arcs are congruent, then the corresponding central angles are congruent.

If we call one of these theorems **direct**, then the other one should be called **converse**.

In this example both theorems, the direct and the converse one, turn out to be true. This is not always the case. For example the theorem: "if two angles are vertical, then they are congruent" is true, but the converse statement: "if two angles are congruent, then they are vertical" is false.

Indeed, suppose that in some angle the bisector is drawn (Figure 13). It divides the angle into two smaller ones. These smaller angles are congruent to each other, but they are not vertical.

EXERCISES

42. Formulate definitions of supplementary angles (§22) and vertical angles (§26) using the notion of *sides* of an angle.

43. Find in the text the definitions of an angle, its vertex and sides, in terms of the notion of a *ray drawn from a point.*

*44.** In Introduction, find the definitions of a ray and a straight segment in terms of the notions of a *straight line* and a point. Are there definitions of a point, line, plane, surface, geometric solid? Why?
Remark: These are examples of geometric notions which are considered **undefinable**.

45. Is the following proposition from §6 a definition, axiom or theorem: "Two segments are congruent if they can be laid one onto the other so that their endpoints coincide"?

46. In the text, find the definitions of a geometric figure, and congruent geometric figures. Are there definitions of congruent segments, congruent arcs, congruent angles? Why?

47. Define a circle.

48. Formulate the proposition converse to the theorem: "If a number is divisible by 2 and by 3, then it is divisible by 6." Is the converse proposition true? Why?

49. In the proposition from §10: "Two arcs of the same circle are congruent if they can be aligned so that their endpoints coincide," separate the hypothesis from the conclusion, and state the converse proposition. Is the converse proposition true? Why?

50. In the theorem: "Bisectors of supplementary angles are perpendicular," separate the hypothesis from the conclusion, and formulate the converse proposition. Is the converse proposition true?

51. Give an example that disproves the proposition: "If the bisectors of two angles with a common vertex are perpendicular, then the angles are supplementary." Is the converse proposition true?

4 Polygons and triangles

31. Broken lines. Straight segments not lying on the same line are said to form a **broken line** (Figures 31, 32) if the endpoint of the first segment is the beginning of the second one, the endpoint of the second segment is the beginning of the third one, and so on. These segments are called **sides**, and the vertices of the angles formed by the adjacent segments **vertices** of the broken line. A broken line is denoted by the row of letters labeling its vertices and endpoints; for instance, one says: "the broken line $ABCDE$."

A broken line is called **convex** if it lies on one side of each of its segments continued indefinitely in both directions. For example, the broken line shown in Figure 31 is convex while the one shown in Figure 32 is not (it lies not on one side of the line BC).

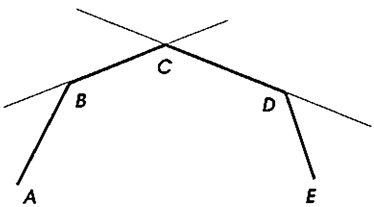

Figure 31

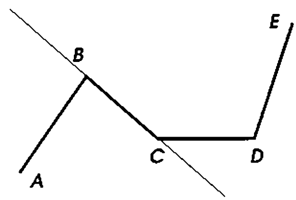

Figure 32

A broken line whose endpoints coincide is called **closed** (e.g. the lines $ABCDE$ or $ADCBE$ in Figure 33). A closed broken line may have self-intersections. For instance, in Figure 33, the line $ADCBE$ is self-intersecting, while $ABCDE$ is not.

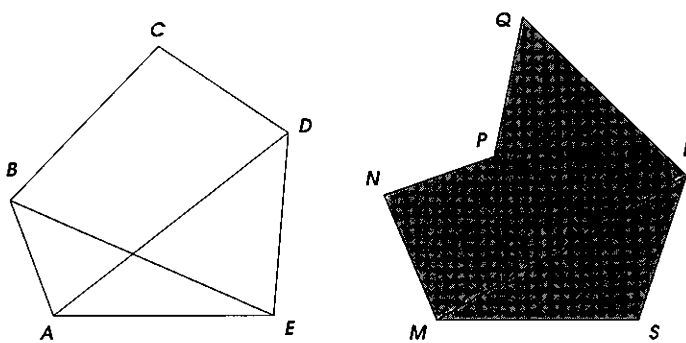

Figure 33

32. Polygons. The figure formed by a non-self-intersecting closed broken line together with the part of the plane bounded by this line is called a **polygon** (Figure 33). The sides and vertices of this broken line are called respectively **sides** and **vertices** of the polygon, and the angles formed by each two adjacent sides **(interior) angles** of the polygon. More precisely, the interior of a polygon's angle is considered that side which contains the interior part of the polygon in the vicinity of the vertex. For instance, the angle at the vertex P of the polygon $MNPQRS$ is the angle greater than $2d$ (with the interior region shaded in Figure 33). The broken line itself is called the **boundary** of the polygon, and the segment congruent to the sum of all of its sides — the **perimeter**. A half of the perimeter is often referred to as the **semiperimeter**.

A polygon is called **convex** if it is bounded by a convex broken line. For example, the polygon $ABCDE$ shown in Figure 33 is convex while the polygon $MNPQRS$ is not. We will mainly consider convex polygons.

Any segment (like AD, BE, MR, ..., Figure 33) which connects two vertices not belonging to the same side of a polygon is called a **diagonal** of the polygon.

The smallest number of sides in a polygon is three. Polygons are named according to the number of their sides: **triangles**, **quadrilaterals**, **pentagons**, **hexagons**, and so on.

The word "triangle" will often be replaced by the symbol $\triangle$.

33. Types of triangles. Triangles are classified by relative lengths of their sides and by the magnitude of their angles. With respect to the lengths of sides, triangles can be **scalene** (Figure 34) — when all three sides have different lengths, **isosceles** (Figure 35) — when two sides are congruent, and **equilateral** (Figure 36) — when all three sides are congruent.

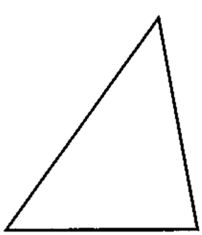

Figure 34

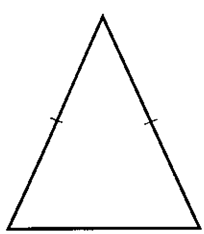

Figure 35

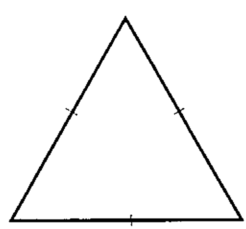

Figure 36

With respect to the magnitude of angles, triangles can be **acute** (Figure 34) — when all three angles are acute, **right** (Figure 37) —

when among the angles there is a right one, and **obtuse** (Figure 38) — when among the angles there is an obtuse one. [3]

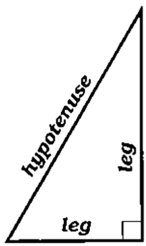

Figure 37

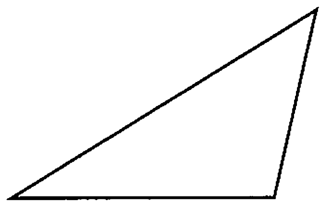

Figure 38

In a right triangle, the sides of the right angle are called **legs**, and the side opposite to the right angle the **hypotenuse**.

34. Important lines in a triangle. One of a triangle's sides is often referred to as **the base**, in which case the opposite vertex is called *the* vertex of the triangle, and the other two sides are called **lateral**. Then the perpendicular dropped from the vertex to the base or to its continuation is called an **altitude**. Thus, if in the triangle ABC (Figure 39), the side AC is taken for the base, then B is the vertex, and BD is the altitude.

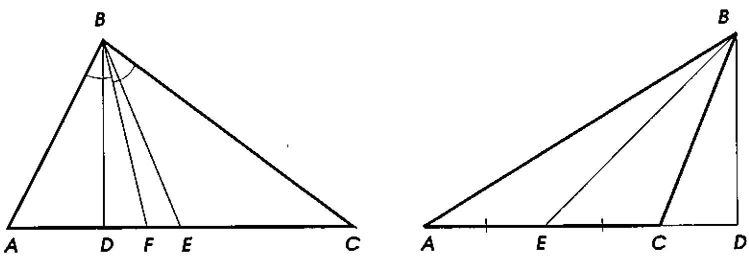

Figure 39

The segment (BE, Figure 39) connecting the vertex of a triangle with the midpoint of the base is called a **median**. The segment (BF) dividing the angle at the vertex into halves is called a **bisector** of the triangle (which generally speaking differs from both the median and the altitude).

[3]We will see in §43 that a triangle may have at most one right or obtuse angle.

Any triangle has three altitudes, three medians, and three bisectors, since each side of the triangle can take on the role of the base.

In an isosceles triangle, usually the side other than each of the two congruent ones is called the base. Respectively, the vertex of an isosceles triangle is the vertex of that angle which is formed by the congruent sides.

EXERCISES

52. Four points on the plane are vertices of three different quadrilaterals. How can this happen?

53. Can a convex broken line self-intersect?

54. Is it possible to tile the entire plane by non-overlapping polygons all of whose angles contain 140° each?

55. Prove that each diagonal of a quadrilateral either lies entirely in its interior, or entirely in its exterior. Give an example of a pentagon for which this is false.

56. Prove that a closed convex broken line is the boundary of a polygon.

57. Is an equilateral triangle considered isosceles? Is an isosceles triangle considered scalene?

*58.** How many intersection points can three straight lines have?

59. Prove that in a right triangle, three altitudes pass through a common point.

60. Show that in any triangle, every two medians intersect. Is the same true for every two bisectors? altitudes?

61. Give an example of a triangle such that only one of its altitudes lies in its interior.

5 Isosceles triangles and symmetry

35. Theorems.

(1) ***In an isosceles triangle, the bisector of the angle at the vertex is at the same time the median and the altitude.***

(2) ***In an isosceles triangle, the angles at the base are congruent.***

Let $\triangle ABC$ (Figure 40) be isosceles, and let the line BD be the bisector of the angle B at the vertex of the triangle. It is required to prove that this bisector BD is also the median and the altitude.

Imagine that the diagram is folded along the line BD so that $\angle ABD$ falls onto $\angle CBD$. Then, due to congruence of the angles 1 and 2, the side AB will fall onto the side CB, and due to congruence of these sides, the point A will merge with C. Therefore DA will coincide with DC, the angle 3 will coincide with the angle 4, and the angle 5 with 6. Therefore

$$DA = DC, \quad \angle 3 = \angle 4, \quad \text{and} \quad \angle 5 = \angle 6.$$

It follows from $DA = DC$ that BD is the median. It follows from the congruence of the angles 3 and 4 that these angles are right, and hence BD is the altitude of the triangle. Finally, the angles 5 and 6 at the base of the triangle are congruent.

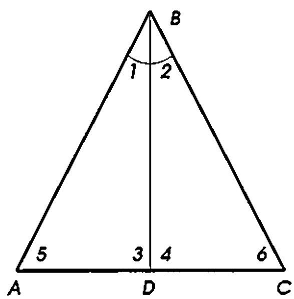

Figure 40

36. Corollary. We see that in the isosceles triangle ABC (Figure 40) the very same line BD possesses four properties: it is the bisector drawn from the vertex, the median to the base, the altitude dropped from the vertex to the base, and finally the perpendicular erected from the base at its midpoint.

Since each of these properties determines the position of the line BD unambiguously, then the validity of any of them implies all the others. For example, *the altitude dropped to the base of an isosceles triangle is at the same time its bisector drawn from the vertex, the median to the base, and the perpendicular erected at its midpoint.*

37. Axial symmetry. If two points (A and A', Figure 41) are situated on the opposite sides of a line a, on the same perpendicular to this line, and the same distance away from the foot of the perpendicular (i.e. if AF is congruent to FA'), then such points are called **symmetric** about the line a.

Two figures (or two parts of the same figure) are called symmetric about a line if for each point of one figure (A, B, C, D, E, ..., Figure 41) the point symmetric to it about this line (A', B', C', D', E', ...) belongs to the other figure, and *vice versa*. A figure is said to have an **axis of symmetry** a if this figure is symmetric to itself about the line a, i.e. if for any point of the figure the symmetric point also belongs to the figure.

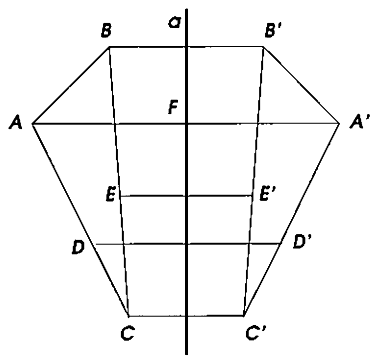

Figure 41

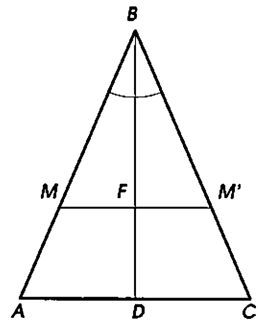

Figure 42

For example, we have seen that the isosceles triangle ABC (Figure 42) is divided by the bisector BD into two triangles (left and right) which can be identified with each other by folding the diagram along the bisector. One can conclude from this that whatever point is taken on the left half of the isosceles triangle, one can always find the point symmetric to it in the right half. For instance, on the side AB, take a point M. Mark on the side BC the segment BM' congruent to BM. We obtain the point M' in the triangle symmetric to M about the axis BD. Indeed, $\triangle MBM'$ is isosceles since $BM = BM'$. Let F denote the intersection point of the segment MM' with the bisector BD of the angle B. Then BK is the bisector in the isosceles triangle MBM'. By §35 it is also the altitude and the median. Therefore MM' is perpendicular to BD, and $MF = M'F$, i.e. M and M' are situated on the opposite sides of BD, on the same perpendicular to BD, and the same distance away from its foot F. Thus ***in an isosceles triangle, the bisector of the angle at the vertex is an axis of symmetry of the triangle.***

38. Remarks. (1) Two symmetric figures can be superimposed by rotating one of them in space about the axis of symmetry until the rotated figure falls into the original plane again. Conversely, if two figures can be identified with each other by turning the plane in space about a line lying in the plane, then these two figures are symmetric about this line.

(2) Although symmetric figures can be superimposed, they are not identical in their position in the plane. This should be understood in the following sense: in order to superimpose two symmetric figures it is *necessary* to flip one of them around and therefore to pull it off the plane temporarily; if however a figure is bound to remain in the plane, no motion can generally speaking identify it with the figure symmetric to it about a line. For example, Figure 43 shows two pairs of symmetric letters: "b" and "d," and "p" and "q." By rotating the letters inside the page one can transform "b" into "q," and "d" into "p," but it is impossible to identify "b" or "q" with "d" or "p" without lifting the symbols off the page.

(3) Axial symmetry is frequently found in nature (Figure 44).

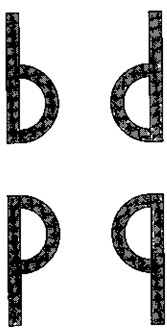

Figure 43

Figure 44

EXERCISES

62. How many axes of symmetry does an equilateral triangle have? How about an isosceles triangle which is not equilateral?

63.* How many axes of symmetry can a quadrilateral have?

64. A kite is a quadrilateral symmetric about a diagonal. Give an example of: (a) a kite; (b) a quadrilateral which is not a kite but has an axis of symmetry.

65. Can a pentagon have an axis of symmetry passing through two (one, none) of its vertices?

66.* Two points A and B are given on the same side of a line MN.

Find a point C on MN such that the line MN would make congruent angles with the sides of the broken line ACB.

Prove theorems:

67. In an isosceles triangle, two medians are congruent, two bisectors are congruent, two altitudes are congruent.

68. If from the midpoint of each of the congruent sides of an isosceles triangle, the segment perpendicular to this side is erected and continued to its intersection with the other of the congruent sides of the triangle, then these two segments are congruent.

69. A line perpendicular to the bisector of an angle cuts off congruent segments on its sides.

70. An equilateral triangle is **equiangular** (i.e. all of its angles are congruent).

71. Vertical angles are symmetric to each other with respect to the bisector of their supplementary angles.

72. A triangle that has two axes of symmetry has three axes of symmetry.

73. A quadrilateral is a kite if it has an axis of symmetry passing through a vertex.

74. Diagonals of a kite are perpendicular.

6 Congruence tests for triangles

39. Preliminaries. As we know, two geometric figures are called congruent if they can be identified with each other by superimposing. Of course, in the identified triangles, all their corresponding elements, such as sides, angles, altitudes, medians and bisectors, are congruent. However, in order to ascertain that two triangles are congruent, there is no need to establish congruence of all their corresponding elements. It suffices only to verify congruence of some of them.

40. Theorems. [4]

(1) **SAS-test:** ***If two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in another triangle, then such triangles are congruent.***

(2) **ASA-test:** ***If one side and two angles adjacent to it in one triangle are congruent respectively to one side and two***

[4]SAS stands for "side–angle–side", ASA for "angle-side-angle, and of course SSS for "side-side-side."

angles adjacent to it in another triangle, then such triangles are congruent.

(3) **SSS-test**: ***If three sides of one triangle are congruent respectively to three sides of another triangle, then such triangles are congruent.***

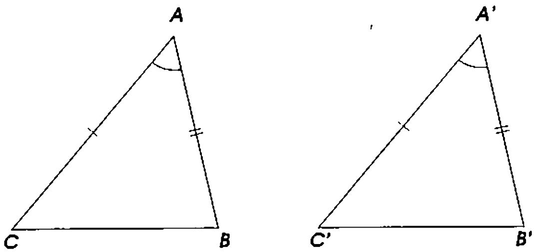

Figure 45

(1) Let ABC and $A'B'C'$ be two triangles (Figure 45) such that

$$AC = A'C', \quad AB = A'B', \quad \angle A = \angle A'.$$

It is required to prove that these triangles are congruent.

Superimpose $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that A would coincide with A', the side AC would go along $A'C'$, and the side AB would lie on the same side of $A'C'$ as $A'B'$.[5] Then: since AC is congruent to $A'C'$, the point C will merge with C''; due to congruence of $\angle A$ and $\angle A'$, the side AB will go along $A'B'$, and due to congruence of these sides, the point B will merge with B'. Therefore the side BC will coincide with $B'C'$ (since two points can be joined by only one line), and hence the entire triangles will be identified with each other. Thus they are congruent.

(2) Let ABC and $A'B'C'$ (Figure 46) be two triangles such that

$$\angle C = \angle C', \quad \angle B = \angle B', \quad CB = C'B'.$$

It is required to prove that these triangles are congruent. Superimpose $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that the point C would coincide with C', the side CB would go along $C'B'$, and the vertex A would lie on the same side of $C'B'$ as A'. Then: since CB is congruent to $C'B'$, the point B will merge with B', and due to congruence of

[5]For this and some other operations in this section it might be necessary to flip the triangle over.

the angles B and B', and C and C', the side BA will go along $B'A'$, and the side CA will go along $C'A'$. Since two lines can intersect only at one point, the vertex A will have to merge with A'. Thus the triangles are identified and are therefore congruent.

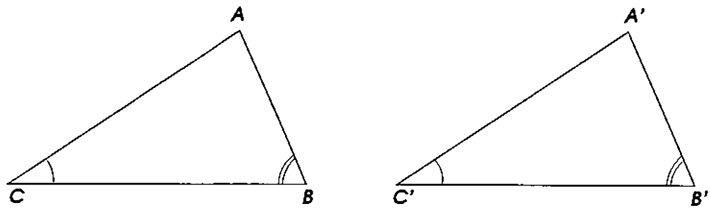

Figure 46

(3) Let ABC and $A'B'C'$ be two triangles such that

$$AB = A'B', \quad BC = B'C', \quad CA = C'A'.$$

It is required to prove that these triangles are congruent. Proving this test by superimposing, the same way as we proved the first two tests, turns out to be awkward, because knowing nothing about the measure of the angles, we would not be able to conclude from coincidence of two corresponding sides that the other sides coincide as well. Instead of superimposing, let us apply *juxtaposing*.

Juxtapose $\triangle ABC$ and $\triangle A'B'C'$ in such a way that their congruent sides AC and $A'C'$ would coincide (i.e. A would merge with A' and C with C'), and the vertices B and B' would lie on the opposite sides of $A'C'$. Then $\triangle ABC$ will occupy the position $\triangle A'B''C'$ (Figure 47). Joining the vertices B' and B'' we obtain two isosceles triangles $B'A'B''$ and $B'C'B''$ with the common base $B'B''$. But in an isosceles triangle, the angles at the base are congruent (§35). Therefore $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, and hence $\angle A'B'C' = \angle A'B''C' = \angle B$. But then the given triangles must be congruent, since two sides and the angle enclosed by them in one triangle are congruent respectively to two sides and the angle enclosed by them in the other triangle.

Remark. In congruent triangles, congruent angles are opposed to congruent sides, and conversely, congruent sides are opposed to congruent angles.

The congruence tests just proved, and the skill of recognizing congruent triangles by the above criteria facilitate solutions to many geometry problems and are necessary in the proofs of many theorems. These congruence tests are the principal means in discovering properties of complex geometric figures. The reader will have many occasions to see this.

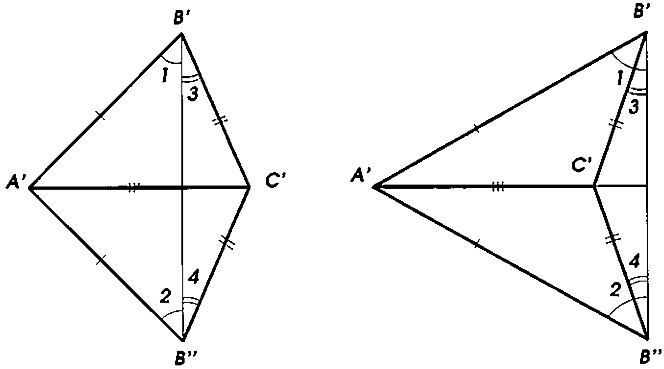

Figure 47

EXERCISES

75. Prove that a triangle that has two congruent angles is isosceles.

76. In a given triangle, an altitude is a bisector. Prove that the triangle is isosceles.

77. In a given triangle, an altitude is a median. Prove that the triangle is isosceles.

78. On each side of an equilateral triangle ABC, congruent segments AB', BC', and AC' are marked, and the points A', B', and C' are connected by lines. Prove that the triangle $A'B'C'$ is also equilateral.

79. Suppose that an angle, its bisector, and one side of this angle in one triangle are respectively congruent to an angle, its bisector, and one side of this angle in another triangle. Prove that such triangles are congruent.

80. Prove that if two sides and the median drawn to the first of them in one triangle are respectively congruent to two sides and the median drawn to the first of them in another triangle, then such triangles are congruent.

81. Give an example of two non-congruent triangles such that two sides and one angle of one triangle are respectively congruent to two sides and one angle of the other triangle.

*82.** On one side of an angle A, the segments AB and AC are marked, and on the other side the segments $AB' = AB$ and $AC' = AC$. Prove that the lines BC' and $B'C$ meet on the bisector of the angle A.

83. Derive from the previous problem a method of constructing the bisector using straightedge and compass.

84. Prove that in a convex pentagon: (a) if all sides are congruent, and all diagonals are congruent, then all interior angles are congruent, and (b) if all sides are congruent, and all interior angles are congruent, then all diagonals are congruent.

85. Is this true that in a convex polygon, if all diagonals are congruent, and all interior angles are congruent, then all sides are congruent?

7 Inequalities in triangles

41. Exterior angles. The angle supplementary to an angle of a triangle (or polygon) is called an **exterior angle** of this triangle (polygon).

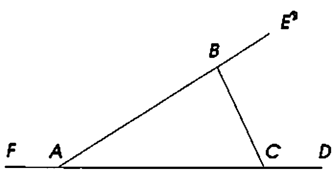

Figure 48

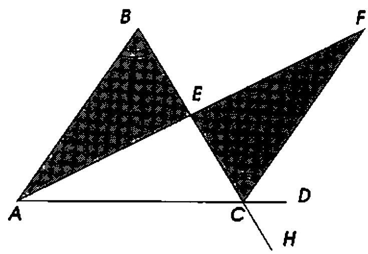

Figure 49

For instance (Figure 48), $\angle BCD$, $\angle CBE$, $\angle BAF$ are exterior angles of the triangle ABC. In contrast with the exterior angles, the angles of the triangle (polygon) are sometimes called **interior**.

For each interior angle of a triangle (or polygon), one can construct two exterior angles (by extending one or the other side of the angle). Such two exterior angles are congruent since they are vertical.

42. Theorem. ***An exterior angle of a triangle is greater than each interior angle not supplementary to it.***

For example, let us prove that the exterior angle BCD of $\triangle ABC$ (Figure 49) is greater than each of the interior angles A and B not supplementary to it.

Through the midpoint E of the side BC, draw the median AE and on the continuation of the median mark the segment EF congruent to AE. The point F will obviously lie in the interior of the angle BCD. Connect F with C by a segment. The triangles ABE and EFC (shaded in Figure 49) are congruent since at the vertex E they have congruent angles enclosed between two respectively congruent sides. From congruence of the triangles we conclude that the angles B and ECF, opposite to the congruent sides AE and EF, are congruent too. But the angle ECF forms a part of the exterior angle BCD and is therefore smaller than $\angle BCD$. Thus the angle B is smaller than the angle BCD.

By continuing the side BC past the point C we obtain the exterior angle ACH congruent to the angle BCD. If from the vertex B, we draw the median to the side AC and double the median by continuing it past the side AC, then we will similarly prove that the angle A is smaller than the angle ACH, i.e. it is smaller than the angle BCD.

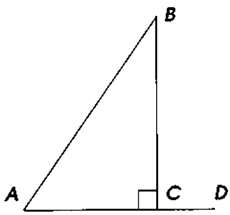

Figure 50

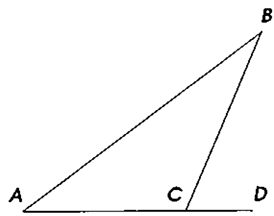

Figure 51

43. Corollary. *If in a triangle one angle is right or obtuse, then the other two angles are acute.*

Indeed, suppose that the angle C in $\triangle ABC$ (Figure 50 or 51) is right or obtuse. Then the supplementary to it exterior angle BCD has to be right or acute. Therefore the angles A and B, which by the theorem are smaller than this exterior angle, must both be acute.

44. Relationships between sides and angles of a triangle.

Theorems. ***In any triangle***

(1) ***the angles opposite to congruent sides are congruent;***

(2) ***the angle opposite to a greater side is greater.***

(1) If two sides of a triangle are congruent, then the triangle is isosceles, and therefore the angles opposite to these sides have to be congruent as the angles at the base of an isosceles triangle (§35).

(2) Let in $\triangle ABC$ (Figure 52) the side AB be greater than BC. It is required to prove that the angle C is greater than the angle A.

On the greater side BA, mark the segment BD congruent to the smaller side BC and draw the line joining D with C. We obtain an isosceles triangle DBC, which has congruent angles at the base, i.e. $\angle BDC = \angle BCD$. But the angle BDC, being an exterior angle with respect to $\triangle ADC$, is greater than the angle A, and hence the angle BCD is also greater than the angle A. Therefore the angle BCA containing $\angle BCD$ as its part is greater than the angle A too.

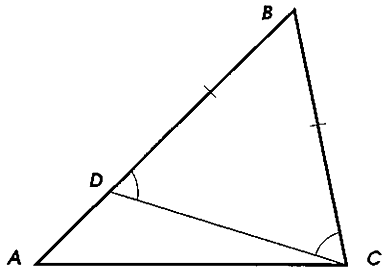

Figure 52

45. The converse theorems. ***In any triangle***

(1) ***the sides opposite to congruent angles are congruent;***

(2) ***the side opposite to a greater angle is greater.***

(1) Let in $\triangle ABC$ the angles A and C be congruent (Figure 53); it is required to prove that $AB = BC$.

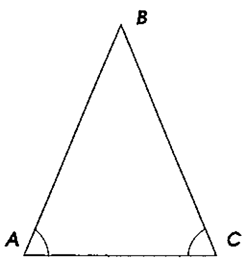

Figure 53

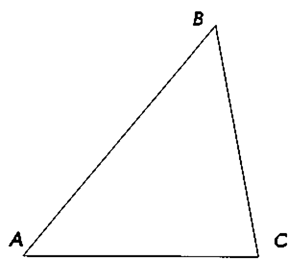

Figure 54

Suppose the contrary is true, i.e. that the sides AB and BC are not congruent. Then one of these sides is greater than the other, and therefore according to the direct theorem, one of the angles A and C has to be greater than the other. But this contradicts the hypothesis that $\angle A = \angle C$. Thus the assumption that AB and BC are non-congruent is impossible. This leaves only the possibility that $AB = BC$.

(2) Let in $\triangle ABC$ (Figure 54) the angle C be greater than the angle A. It is required to prove that $AB > BC$.

Suppose the contrary is true, i.e. that AB is not greater than BC. Then two cases can occur: either $AB = BC$ or $AB < BC$.

According to the direct theorem, in the first case the angle C would have been congruent to the angle A, and in the second case the angle C would have been smaller than the angle A. Either conclusion contradicts the hypothesis, and therefore both cases are excluded. Thus the only remaining possibility is $AB > BC$.

Corollary.

(1) *In an equilateral triangle all angles are congruent.*

(2) *In an equiangular triangle all sides are congruent.*

46. Proof by contradiction. The method we have just used to prove the converse theorems is called **proof by contradiction**, or **reductio ad absurdum**. In the beginning of the argument the assumption contrary to what is required to prove is made. Then by reasoning on the basis of this assumption one arrives at a contradiction (absurd). This result forces one to reject the initial assumption and thus to accept the one that was required to prove. This way of reasoning is frequently used in mathematical proofs.

47. A remark on converse theorems. It is a mistake, not uncommon for beginning geometry students, to assume that the converse theorem is automatically established whenever the validity of a direct theorem has been verified. Hence the false impression that proof of converse theorems is unnecessary at all. As it can be shown by examples, like the one given in §30, this conclusion is erroneous. Therefore converse theorems, when they are valid, require separate proofs.

However, in the case of congruence or non-congruence of two sides of a triangle ABC, e.g. the sides AB and BC, only the following three cases can occur:

$$AB = BC, \quad AB > BC, \quad AB < BC.$$

Each of these three cases excludes the other two: say, if the first case $AB = BC$ takes place, then neither the 2nd nor the 3rd case is possible. In the theorem of §44, we have considered all the three cases and arrived at the following respective conclusions regarding the opposite angles C and A:

$$\angle C = \angle A, \quad \angle C > \angle A, \quad \angle C < \angle A.$$

Each of these conclusions excludes the other two. We have also seen in §45 that the converse theorems are true and can be easily proved by *reductio ad absurdum.*

In general, if in a theorem, or several theorems, we address all possible mutually exclusive cases (which can occur regarding the magnitude of a certain quantity or disposition of certain parts of a figure), and it turns out that in these cases we arrive at mutually exclusive conclusions (regarding some other quantities or parts of the figure), then we can claim *a priori* that the converse propositions also hold true.

We will encounter this rule of convertibility quite often.

48. Theorem. ***In a triangle, each side is smaller than the sum of the other two sides.***

If we take a side which is not the greatest one in a triangle, then of course it will be smaller than the sum of the other two sides. Therefore we need to prove that even the greatest side of a triangle is smaller than the sum of the other two sides.

In $\triangle ABC$ (Figure 55), let the greatest side be AC. Continuing the side AB past B mark on it the segment $BD = BC$ and draw DC. Since $\triangle BDC$ is isosceles, then $\angle D = \angle DCB$. Therefore the angle D is smaller than the angle DCA, and hence in $\triangle ADC$ the side AC is smaller than AD (§45), i.e. $AC < AB + BD$. Replacing BD with BC we get

$$AC < AB + BC.$$

Corollary. From both sides of the obtained inequality, subtract AB or BC:

$$AC - AB < BC, \quad AC - BC < AB.$$

Reading these inequalities from right to left we see that each of the sides BC and AB is greater than the difference of the other two sides. Obviously, the same can also be said about the greatest side AC, and therefore *in a triangle, each side is greater than the difference of the other two sides.*

Remarks. (1) The inequality described in the theorem is often called the **triangle inequality**.

(2) When the point B lies on the segment AC, the triangle inequality turns into the equality $AC = AB + BC$. More generally, if three points lie on the same line (and thus do not form a triangle), then the greatest of the three segments connecting these points is the sum of the other two segments. Therefore ***for any three points*** it is

still true that ***the segment connecting two of them is smaller than or congruent to the sum of the other two segments.***

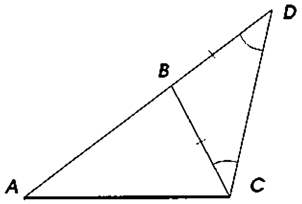

Figure 55

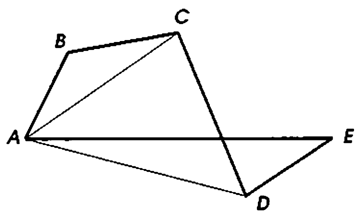

Figure 56

49. Theorem. ***The line segment connecting any two points is smaller than any broken line connecting these points.***

If the broken line in question consists of only two sides, then the theorem has already been proved in §48. Consider the case when the broken line consists of more than two sides. Let AE (Figure 56) be the line segment connecting the points A and E, and let $ABCDE$ be a broken line connecting the same points. We are required to prove that AE is smaller than the sum $AB + BC + CD + DE$.

Connecting A with C and D and using the triangle inequality we find:

$$AE \leq AD + DE, \quad AD \leq AC + CD, \quad AC \leq AB + BC.$$

Moreover, these inequalities cannot turn into equalities all at once. Indeed, if this happened, then (Figure 57) D would lie on the segment AE, C on AD, B on AB, i.e. $ABCDE$ would not be a broken line, but the straight segment AE. Thus adding the inequalities termwise

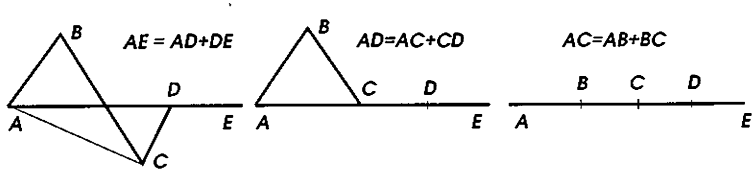

Figure 57

and subtracting AD and AC from both sides we get

$$AE < AB + BC + CD + DE.$$

50. Theorem. ***If two sides of one triangle are congruent respectively to two sides of another triangle, then:***

(1) ***the greater angle contained by these sides is opposed to the greater side;***

(2) *vice versa,* ***the greater of the non-congruent sides is opposed to the greater angle.***

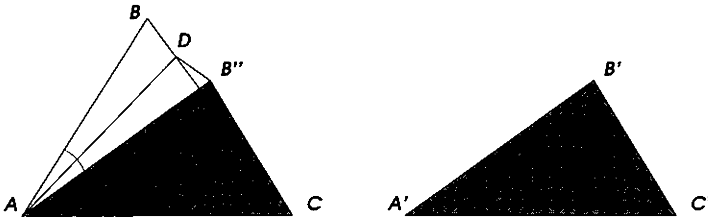

Figure 58

(1) In $\triangle ABC$ and $\triangle A'B'C'$, we are given:

$$AB = A'B', \quad AC = A'C', \quad \angle A > \angle A'.$$

We are required to prove that $BC > B'C'$. Put $\triangle A'B'C'$ onto $\triangle ABC$ in a way (shown in Figure 58) such that the side $A'C'$ would coincide with AC. Since $\angle A' < \angle A$, then the side $A'B'$ will lie inside the angle A. Let $\triangle A'B'C'$ occupy the position $AB''C$ (the vertex B'' may fall outside or inside of $\triangle ABC$, or on the side BC, but the forthcoming argument applies to all these cases). Draw the bisector AD of the angle BAB'' and connect D with B''. Then we obtain two triangles ABD and DAB'' which are congruent because they have a common side AD, $AB = AB''$ by hypothesis, and $\angle BAD = \angle BAD''$ by construction. Congruence of the triangles implies $BD = DB''$. From $\triangle DCB''$ we now derive: $B''C < B''D + DC$ (§48). Replacing $B''D$ with BD we get

$$B''C < BD + DC, \quad \text{and hence} \quad B'C' < BC.$$

(2) Suppose in the same triangles ABC and $A'B'C'$ we are given that $AB = A'B'$, $AC = A'C'$ and $BC > B'C'$; let us prove that $\angle A > \angle A'$.

Assume the contrary, i.e. that the $\angle A$ is not greater than $\angle A'$. Then two cases can occur: either $\angle A = \angle A'$ or $\angle A < \angle A'$. In the first case the triangles would have been congruent (by the SAS-test) and therefore the side BC would have been congruent to $B'C'$, which contradicts the hypotheses. In the second case the side BC would have been smaller than $B'C'$ by part (1) of the theorem, which contradicts the hypotheses too. Thus both of these cases are excluded; the only case that remains possible is $\angle A > \angle A'$.

EXERCISES

86. Can an exterior angle of an isosceles triangle be smaller than the supplementary interior angle? Consider the cases when the angle is: (a) at the base, and (b) at the vertex.

87. Can a triangle have sides: (a) 1, 2, and 3 *cm* (centimeters) long? (b) 2, 3, and 4 *cm* long?

88. Can a quadrilateral have sides: 2, 3, 4, and 10 *cm* long?

`Prove theorems:`

89. A side of a triangle is smaller than its semiperimeter.

90. A median of a triangle is smaller than its semiperimeter.

91.* A median drawn to a side of a triangle is smaller than the semisum of the other two sides.
`Hint`: Double the median by prolonging it past the midpoint of the first side.

92. The sum of the medians of a triangle is smaller than its perimeter but greater than its semi-perimeter.

93. The sum of the diagonals of a quadrilateral is smaller than its perimeter but greater than its semi-perimeter.

94. The sum of segments connecting a point inside a triangle with its vertices is smaller than the semiperimeter of the triangle.

95.* Given an acute angle XOY and an interior point A. Find a point B on the side OX and a point C on the side OY such that the perimeter of the triangle ABC is minimal.
`Hint`: Introduce points symmetric to A with respect to the sides of the angle.

8 Right triangles

51. Comparative length of the perpendicular and a slant.

`Theorem.` ***The perpendicular dropped from any point to a line is smaller than any slant drawn from the same point to this line.***

Let AB (Figure 59) be the perpendicular dropped from a point A to the line MN, and AC be any slant drawn from the same point A to the line MN. It is required to show that $AB < AC$.

In $\triangle ABC$, the angle B is right, and the angle C is acute (§43). Therefore $\angle C < \angle B$, and hence $AB < AC$, as required.

Remark. By "the distance from a point to a line," one means the *shortest* distance which is measured along the perpendicular dropped from this point to the line.

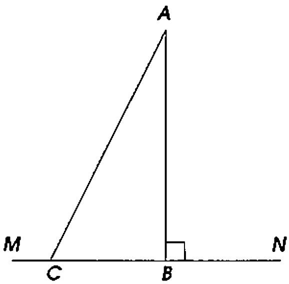

Figure 59

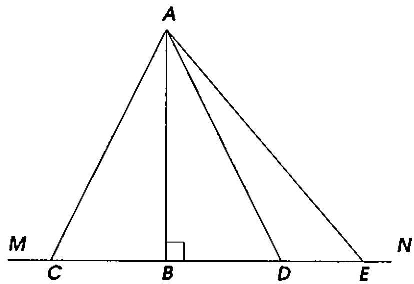

Figure 60

52. Theorem. ***If the perpendicular and some slants are drawn to a line from the same point outside this line, then:***

(1) ***if the feet of the slants are the same distance away from the foot of the perpendicular, then such slants are congruent;***

(2) ***if the feet of two slants are not the same distance away from the foot of the perpendicular, then the slant whose foot is farther away from the foot of the perpendicular is greater.***

(1) Let AC and AD (Figure 60) be two slants drawn from a point A to the line MN and such that their feet C and D are the same distance away from the foot B of the perpendicular AB, i.e. $CB = BD$. It is required to prove that $AC = AD$.

In the triangles ABC and ABD, AB is a common side, and beside this $BC = BD$ (by hypothesis) and $\angle ABC = \angle ABD$ (as right angles). Therefore these triangles are congruent, and thus $AC = AD$.

(2) Let AC and AE (Figure 59) be two slants drawn from the point A to the line MN and such that their feet are not the same distance away from the foot of the perpendicular; for instance, let $BE > BC$. It is required to prove that $AE > AC$.

Mark $BD = BC$ and draw AD. By part (1), $AD = AC$. Compare AE with AD. The angle ADE is exterior with respect to $\triangle ABD$ and therefore it is greater than the right angle. Therefore the angle ADE is obtuse, and hence the angle AED must be acute (§43). It follows that $\angle ADE > \angle AED$, therefore $AE > AD$, and thus $AE > AC$.

53. The converse theorems. ***If some slants and the perpendicular are drawn to a line from the same point outside this line, then:***

(1) ***if two slants are congruent, then their feet are the same distance away from the foot of the perpendicular;***

(2) ***if two slants are not congruent, then the foot of the greater one is farther away from the foot of the perpendicular.***

We leave it to the readers to prove these theorems (by the method of *reductio ad absurdum*).

54. Congruence tests for right triangles. Since in right triangles the angles contained by the legs are always congruent as right angles, then ***right triangles are congruent:***

(1) *if the legs of one of them are congruent respectively to the legs of the other;*

(2) *if a leg and the acute angle adjacent to it in one triangle are congruent respectively to a leg and the acute angle adjacent to it in the other triangle.*

These two tests require no special proof, since they are particular cases of the general SAS- and ASA-tests. Let us prove the following two tests which apply to right triangles only.

55. Two tests requiring special proofs.

Theorems. ***Two right triangles are congruent:***

(1) ***if the hypotenuse and an acute angle of one triangle are congruent to respectively the hypotenuse and an acute angle of the other.***

(2) ***if the hypotenuse and a leg of one triangle are congruent respectively to the hypotenuse and a leg of the other.***

(1) Let ABC and $A_1B_1C_1$ (Figure 61) be two right triangles such that $AB = A_1B_1$ and $\angle A = \angle A_1$. It is required to prove that these triangles are congruent.

Put $\triangle ABC$ onto $\triangle A_1B_1C_1$ in a way such that their congruent hypotenuses coincide. By congruence of the angles A and A_1, the leg AC will go along A_1C_1. Then, if we assume that the point C

occupies a position C_2 or C_3 different from C_1, we will have two perpendiculars (B_1C_1 and B_1C_2, or B_1C_1 and B_1C_3) dropped from the same point B' to the line $A'C'$. Since this is impossible (§24), we conclude that the point C will merge with C_1.

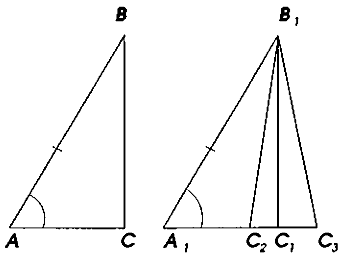

Figure 61

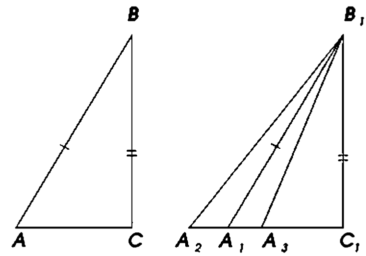

Figure 62

(2) Let (Figure 62), in the right triangles, it be given: $AB = A_1B_1$ and $BC = B_1C_1$. It is required to prove that the triangles are congruent. Put $\triangle ABC$ onto $\triangle A_1B_1C_1$ in a way such that their congruent legs BC and B_1C_1 coincide. By congruence of right angles, the side CA will go along C_1A_1. Then, if we assume that the hypotenuse AB occupies a position A_2B_1 or A_3B_1 different from A_1B_1, we will have two congruent slants (A_1B_1 and A_2B_1, or A_1B_1 and A_3B_1) whose feet are not the same distance away from the foot of the perpendicular B_1C_1. Since this is impossible (§53) we conclude that AB will be identified with A_1B_1.

EXERCISES

Prove theorems:

96. Each leg of a right triangle is smaller than the hypotenuse.

97. A right triangle can have at most one axis of symmetry.

98. At most two congruent slants to a given line can be drawn from a given point.

99.* Two isosceles triangles with a common vertex and congruent lateral sides cannot fit one inside the other.

100. The bisector of an angle is its axis of symmetry.

101. A triangle is isosceles if two of its altitudes are congruent.

102. A median in a triangle is equidistant from the two vertices not lying on it.

103.* A line and a circle can have at most two common points.

9 Segment and angle bisectors

56. The perpendicular bisector, i.e. the perpendicular to a segment erected at the midpoint of the segment, and the bisector of an angle have very similar properties. To see the resemblance better we will describe the properties in a parallel fashion.

(1) ***If a point*** (K, Figure 63) ***lies on the perpendicular*** (MN) ***erected at the midpoint of a segment*** (AB), ***then the point is the same distance away from the endpoints of the segment*** (i.e. $KA = KB$).

Since $MN \perp AB$ and $AO = OB$, AK and KB are slants to AB, and their feet are the same distance away from the foot of the perpendicular. Therefore $KA = KB$.

(1) ***If a point*** (K, Figure 64) ***lies on the bisector*** (OM) ***of an angle*** (AOB), ***then the point is the same distance away from the sides of the angle*** (i.e. the perpendiculars KD and KC are congruent).

Since OM bisects the angle, the right triangles OCK and ODK are congruent, as they have the common hypotenuse and congruent acute angles at the vertex O. Therefore $KC = KD$.

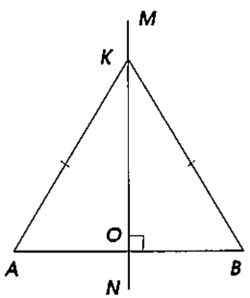

Figure 63

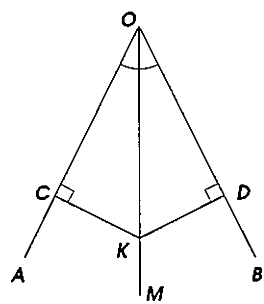

Figure 64

(2) The converse theorem. ***If a point*** (K, Figure 63) ***is the same distance away from the endpoints of the segment*** AB (i.e. if $KA = KB$), ***then the point lies on the perpendicular to*** AB ***passing through its midpoint.***

(2) The converse theorem. ***If an interior point of an angle*** (K, Figure 64) ***is the same distance away from its sides*** (i.e. if the perpendiculars KC and KD are congruent) ***then it lies on the bisector of this angle.***

Through K, draw the line $MN \perp AB$. We get two right triangles KAO and KBO which are congruent as having congruent hypotenuses and the common leg KO. Therefore the line MN drawn through K to be perpendicular to AB bisects it.

Through O and K, draw the line OM. Then we get two right triangles OCK and ODK which are congruent as having the common hypotenuse and the congruent legs CK and DK. Hence they have congruent angles at the vertex O, and therefore the line OM drawn to pass through K bisects the angle AOB.

57. Corollary. From the two proven theorems (direct and converse) one can also derive the following theorems:

If a point does not lie on the perpendicular erected at the midpoint of a segment then the point is unequal distances away from the endpoints of this segment.

If an interior point of an angle does not lie on the ray bisecting it, then the point is unequal distances away from the sides of this angle.

We leave it to the readers to prove these theorems (using the method *reductio ad absurdum*).

58. Geometric locus. The **geometric locus** of points satisfying a certain condition is the curve (or the surface in the space) or, more generally, the set of points, which contains all the points satisfying this condition and contains no points which do not satisfy it.

For instance, the geometric locus of points at a given distance r from a given point C is the circle of radius r with the center at the point C. As it follows from the theorems of §56, §57:

The geometric locus of points equidistant from two given points is the perpendicular to the segment connecting these points, passing through the midpoint of the segment.

The geometric locus of interior points of an angle equidistant from its sides is the bisector of this angle.

59. The inverse theorem. If the hypothesis and the conclusion of a theorem are the *negations* of the hypothesis and the conclusion of another theorem, then the former theorem is called **inverse** to the latter one. For instance, the theorem inverse to: "if the digit sum is divisible by 9, then the number is divisible by 9" is: "if the digit sum is not divisible by 9, then the number is not divisible by 9."

It is worth mentioning that the validity of a direct theorem does not guarantee the validity of the inverse one: for example, the inverse proposition "if not every summand is divisible by a certain number then the sum is not divisible by this number" is false while the direct proposition is true.

The theorem described in §57 (both for the segment and for the angle) is inverse to the (direct) theorem described in §56.

60. Relationships between the theorems: direct, converse, inverse, and contrapositive. For better understanding of the relationship let us denote the hypothesis of the direct theorem by the letter A, and the conclusion by the letter B, and express the theorems concisely as:

(1) **Direct theorem**: if A is true, then B is true;

(2) **Converse theorem**: if B is true, then A is true;

(3) **Inverse theorem**: if A is false, then B is false;

(4) **Contrapositive theorem**: if B is false, then A is false.

Considering these propositions it is not hard to notice that the first one is in the same relationship to the fourth as the second one to the third. Namely, the propositions (1) and (4) can be transformed into each other, and so can the propositions (2) and (3). Indeed, from the proposition: "if A is true, then B is true" it follows immediately that "if B is false, then A is false" (since if A were true, then by the first proposition B would have been true too); and *vice versa*, from the proposition: "if B is false, then A is false" we derive: "if A is true, then B is true" (since if B were false, then A would have been false as well). Quite similarly, we can check that the second proposition follows from the third one, and *vice versa*.

Thus in order to make sure that all the four theorems are valid, there is no need to prove each of them separately, but it suffices to prove only two of them: direct and converse, or direct and inverse.

EXERCISES

104. Prove as a direct theorem that a point not lying on the perpendicular bisector of a segment is not equidistant from the endpoints of the segment; namely it is closer to that endpoint which lies on the same side of the bisector.

105. Prove as a direct theorem that any interior point of an angle which does not lie on the bisector is not equidistant from the sides of the angle.

106. Prove that two perpendiculars to the sides of an angle erected at equal distances from the vertex meet on the bisector.

107. Prove that if A and A', and B and B' are two pairs of points symmetric about some line XY, then the four points A, A', B', B lie on the same circle.

108. Find the geometric locus of vertices of isosceles triangles with a given base.

109. Find the geometric locus of the vertices A of triangles ABC with the given base BC and such that $\angle B > \angle C$.

110. Find the geometric locus of points equidistant from two given intersecting infinite straight lines.

111.* Find the geometric locus of points equidistant from three given infinite straight lines, intersecting pairwise.

112. For theorems from §60: direct, converse, inverse, and contrapositive, compare in which of the following four cases each of them is true: when (a) A is true and B is true, (b) A is true but B is false, (c) A is false but B is true, and (d) A is false and B is false.

113. By definition, the **negation** of a proposition is true whenever the proposition is false, and false whenever the proposition is true. State the negation of the proposition: "the digit sum of every multiple of 3 is divisible by 9." Is this proposition true? Is its negation true?

114. Formulate affirmatively the negations of the propositions: (a) in every quadrilateral, both diagonals lie inside it; (b) in every quadrilateral, there is a diagonal that lies inside it; (c) there is a quadrilateral whose both diagonals lie inside it; (d) there is a quadrilateral that has a diagonal lying outside it. Which of these propositions are true?

10 Basic construction problems

61. Preliminary remarks. Theorems we proved earlier allow us to solve some **construction** problems. Note that in elementary geometry one considers those constructions which can be performed using only *straightedge and compass.* [6]

62. Problem 1. ***To construct a triangle with the given three sides*** a, b ***and*** c (Figure 65).

[6] As we will see, the use of the drafting triangle, which can be allowed for saving time in the actual construction, is unnecessary in principle.

On any line MN, mark the segment CB congruent to one of the given sides, say, a. Describe two arcs centered at the points C and B of radii congruent to b and to c. Connect the point A, where these arcs intersect, with B and with C. The required triangle is ABC.

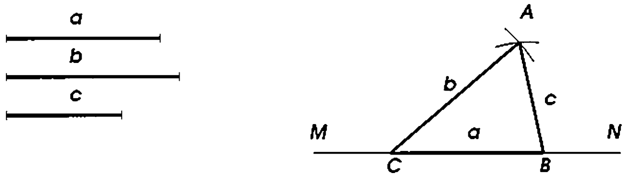

Figure 65

Remark. For three segments to serve as sides of a triangle, it is necessary that the greatest one is smaller than the sum of the other two (§48).

63. Problem 2. ***To construct an angle congruent to the given angle*** ABC ***and such that one of the sides is a given line*** MN, ***and the vertex is at a point*** O ***given on the line*** (Figure 66).

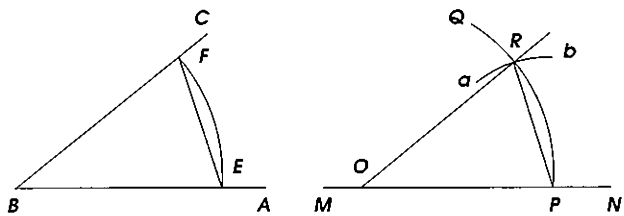

Figure 66

Between the sides of the given angle, describe an arc EF of any radius centered at the vertex B, then keeping the same setting of the compass place its pin leg at the point O and describe an arc PQ. Furthermore, describe an arc ab centered at the point P with the radius equal to the distance between the points E and F. Finally draw a line through O and the point R (the intersection of the two arcs). The angle ROP is congruent to the angle ABC because the triangles ROP and FBE are congruent as having congruent respective sides.

64. Problem 3. ***To bisect a given angle*** (Figure 67), or in other words, ***to construct the bisector of a given angle*** or ***to draw its axis of symmetry.***

Between the sides of the angle, draw an arc DE of arbitrary radius centered at the vertex B. Then, setting the compass to an arbitrary radius, greater however than half the distance between D and E (see Remark to Problem 1), describe two arcs centered at D and E so that they intersect at some point F. Drawing the line BF we obtain the bisector of the angle ABC.

For the proof, connect the point F with D and E by segments. We obtain two triangles BEF and BDF which are congruent since BF is their common side, and $BD = BE$ and $DE = EF$ by construction. The congruence of the triangles implies: $\angle ABF = \angle CBF$.

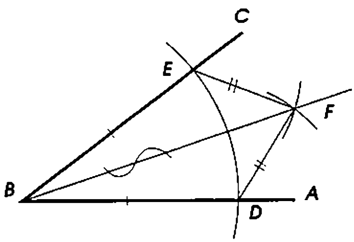

Figure 67

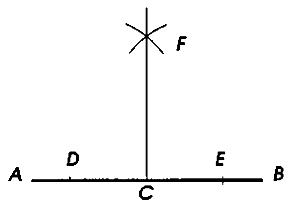

Figure 68

65. Problem 4. ***From a given point C on the line AB, to erect a perpendicular to this line*** (Figure 68).

On both sides of the point C on the line AB, mark congruent segments CD and CE (of any length). Describe two arcs centered at D and E of the same radius (greater than CD) so that the arcs intersect at a point F. The line passing through the points C and F will be the required perpendicular.

Indeed, as it is evident from the construction, the point F will have the same distance from the points D and E; therefore it will lie on the perpendicular to the segment AB passing through its midpoint (§56). Since the midpoint is C, and there is only one line passing through C and F, then $FC \perp DE$.

66. Problem 5. ***From a given point A, to drop a perpendicular to a given line BC*** (Figure 69).

Draw an arc of arbitrary radius (greater however than the distance from A to BC) with the center at A so that it intersects BC at some points D and E. With these points as centers, draw two arcs of the same arbitrary radius (greater however than $\frac{1}{2}DE$) so that they intersect at some point F. The line AF is the required perpendicular.

Indeed, as it is evident from the construction, each of the points A and F is equidistant from D and E, and all such points lie on the perpendicular to the segment AB passing through its midpoint (§58).

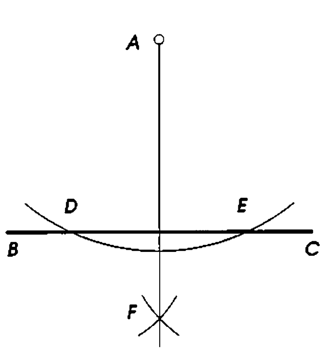

Figure 69

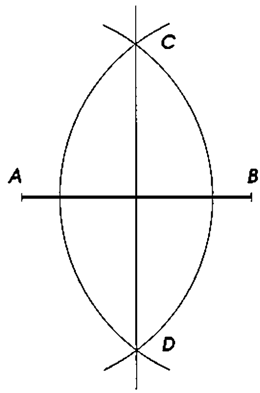

Figure 70

67. Problem 6. ***To draw the perpendicular to a given segment AB through its midpoint*** (Figure 70); in other words, ***to construct the axis of symmetry of the segment AB.***

Draw two arcs of the same arbitrary radius (greater than $\frac{1}{2}AB$), centered at A and B, so that they intersect each other at some points C and D. The line CD is the required perpendicular.

Indeed, as it is evident from the construction, each of the points C and D is equidistant from A and B, and therefore must lie on the symmetry axis of the segment AB.

Problem 7. ***To bisect a given straight segment*** (Figure 70). It is solved the same way as the previous problem.

68. Example of a more complex problem. The basic constructions allow one to solve more complicated construction problems. As an illustration, consider the following problem.

Problem. *To construct a triangle with a given base b, an angle α at the base, and the sum s of the other two sides* (Figure 71). To work out a solution plan, suppose that the problem has been solved, i.e. that a triangle ABC has been found such that the base $AC = b$, $\angle A = \alpha$ and $AB + BC = s$. Examine the obtained diagram. We know how to construct the side AC congruent to b and the angle A congruent to α. Therefore it remains on the other side of the angle to find a point B such that the sum $AB + BC$ is congruent to s.

Continuing AB past B, mark the segment AD congruent to s. Now the problem reduces to finding on AD a point B which would be the same distance away from C and D. As we know (§58), such a point must lie on the perpendicular to CD passing through its midpoint. The point will be found at the intersection of this perpendicular with AD.

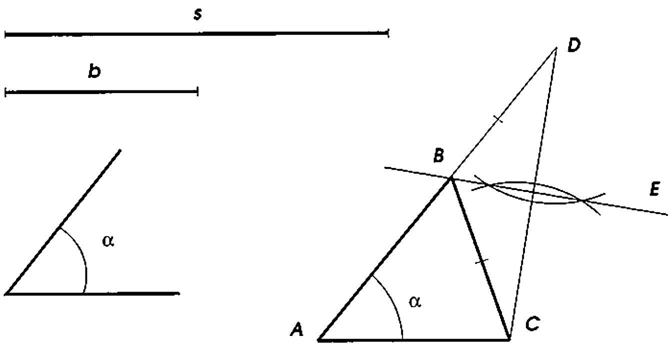

Figure 71

Thus, here is the solution of the problem: construct (Figure 71) the angle A congruent to α. On its sides, mark the segments $AC = b$ and $AD = s$, and connect the point D with C. Through the midpoint of CD, construct the perpendicular BE. Connect its intersection with AD, i.e. the point B, with C. The triangle ABC is a solution of the problem since $AC = b$, $\angle A = \alpha$ and $AB + BC = s$ (because $BD = BC$).

Examining the construction we notice that it is not always possible. Indeed, if the sum s is too small compared to b, then the perpendicular EB may miss the segment AD (or intersect the continuation of AD past A or past D). In this case the construction turns out *impossible*. Moreover, independently of the construction procedure, one can see that the problem has no solution if $s < b$ or $s = b$, because there is no triangle in which the sum of two sides is smaller than or congruent to the third side.

In the case when a solution exists, it turns out to be unique, i.e. there exists only one triangle, [7] satisfying the requirements of the

[7]There are infinitely many triangles satisfying the requirements of the problem, but they are all congruent to each other, and so it is customary to say that the solution of the problem is unique.

problem, since the perpendicular BE can intersect AD at one point at most.

69. Remark. The previous example shows that solution of a complex construction problem should consist of the following four stages.

(1) Assuming that the problem has been solved, we can draft the diagram of the required figure and, carefully examining it, try to find those relationships between the given and required data that would allow one to reduce the problem to other, previously solved problems. This most important stage, whose aim is to work out a plan of the solution, is called **analysis**.

(2) Once a plan has been found, the **construction** following it can be executed.

(3) Next, to validate the plan, one shows on the basis of known theorems that the constructed figure does satisfy the requirements of the problem. This stage is called **synthesis**.

(4) Then we ask ourselves: if the problem has a solution for any given data, if a solution is unique or there are several ones, are there any special cases when the construction simplifies or, on the contrary, requires additional examination. This solution stage is called **research**.

When a problem is very simple, and there is no doubt about possibility of the solution, then one usually omits the analysis and research stages, and provides only the construction and the proof. This was what we did describing our solutions of the first seven problems of this section; this is what we are going to do later on whenever the problems at hand will not be too complex.

EXERCISES

Construct:

115. The sum of two, three, or more given angles.

116. The difference of two angles.

117. Two angles whose sum and difference are given.

118. Divide an angle into 4, 8, 16 congruent parts.

119. A line in the exterior of a given angle passing through its vertex and such that it would form congruent angles with the sides of this angle.

120. A triangle: (a) given two sides and the angle between them; (b) given one side and both angles adjacent to it; (c) given two sides and the angle opposite to the greater one of them; (d) given two sides and the angle opposite to the smaller one of them (in this case there can be two solutions, or one, or none).

121. An isosceles triangle: (a) given its base and another side; (b) given its base and a base angle; (c) given its base angle and the opposite side.

122. A right triangle: (a) given both of its legs; (b) given one of the legs and the hypotenuse; (c) given one of the legs and the adjacent acute angle.

123. An isosceles triangle: (a) given the altitude to the base and one of the congruent sides; (b) given the altitude to the base and the angle at the vertex; (c) given the base and the altitude to another side.

124. A right triangle, given an acute angle and the hypotenuse.

125. Through an interior point of an angle, construct a line that cuts off congruent segments on the sides of the angle.

126. Through an exterior point of an angle, construct a line which would cut off congruent segments on the sides of the angle.

127. Find two segments whose sum and difference are given.

128. Divide a given segment into 4, 8, 16 congruent parts.

129. On a given line, find a point equidistant from two given points (outside the line).

130. Find a point equidistant from the three vertices of a given triangle.

131. On a given line intersecting the sides of a given angle, find a point equidistant from the sides of the angle.

132. Find a point equidistant from the three sides of a given triangle.

133. On an infinite line AB, find a point C such that the rays CM and CN connecting C with two given points M and N situated on the same side of AB would form congruent angles with the rays CA and CB respectively.

134. Construct a right triangle, given one of its legs and the sum of the other leg with the hypotenuse.

135. Construct a triangle, given its base, one of the angles adjacent to the base, and the difference of the other two sides (consider two cases: (1) when the smaller of the two angles adjacent to the base is given; (2) when the greater one is given).

136. Construct a right triangle, given one of its legs and the difference of the other two sides.

137. Given an angle A and two points B and C situated one on one side of the angle and one on the other, find: (1) a point M equidistant from the sides of the angle and such that $MB = MC$; (2) a point N equidistant from the sides of the angle and such that $NB = BC$; (3) a point P such that each of the points B and C would be the same distance away from A and P.

138. Two towns are situated near a straight railroad line. Find the position for a railroad station so that it is equidistant from the towns.

139. Given a point A on one of the sides of an angle B. On the other side of the angle, find a point C such that the sum $CA + CB$ is congruent to a given segment.

11 Parallel lines

70. Definitions. Two lines are called **parallel** if they lie in the same plane and do not intersect one another no matter how far they are extended in both directions.

In writing, parallel lined are denoted by the symbol $\|$. Thus, if two lines AB and CD are parallel, one writes $AB \| CD$.

Existence of parallel lines is established by the following theorem.

71. Theorem. ***Two perpendiculars*** (AB and CD, Figure 72) ***to the same line*** (MN) ***cannot intersect no matter how far they are extended.***

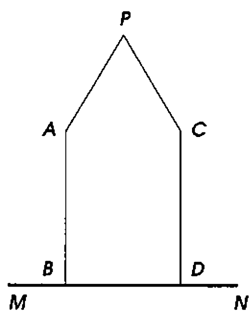

Figure 72

Indeed, if such perpendiculars could intersect at some point P, then two perpendiculars to the line MN would be dropped from this point, which is impossible (§24). Thus two perpendiculars to the same line are parallel to each other.

72. Names of angles formed by intersection of two lines by a transversal. Let two lines AB and CD (Figure 73) be intersected by a third line MN. Then 8 angles are formed (we labeled them by numerals) which carry pairwise the following names:

corresponding angles: 1 and 5, 4 and 8, 2 and 6, 3 and 7;

alternate angles: 3 and 5, 4 and 6 (interior); 1 and 7, 2 and 8 (exterior);

same-side angles: 4 and 5, 3 and 6 (interior); 1 and 8, 2 and 7 (exterior).

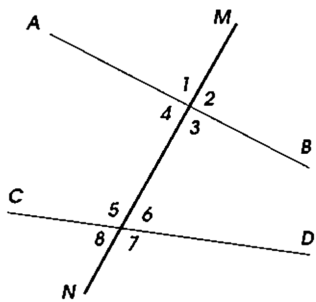

Figure 73

73. Tests for parallel lines. ***When two lines*** (AB and CD, Figure 74) ***are intersected by a third line*** (MN), ***and it turns out that:***

(1) ***some corresponding angles are congruent,*** or

(2) ***some alternate angles are congruent,*** or

(3) ***the sum of some same-side interior or same-side exterior angles is*** $2d$,

then these two lines are parallel.

Suppose, for example, that the corresponding angles 2 and 6 are congruent. We are required to show that in this case $AB \| CD$. Let us assume the contrary, i.e. that the lines AB and CD are not parallel. Then these lines intersect at some point P lying on the right of MN or at some point P' lying on the left of MN. If the intersection is at P, then a triangle is formed for which the angle 2 is exterior, and the angle 6 interior not supplementary to it. Therefore the angle 2 has to be greater than the angle 6 (§42), which contradicts the hypothesis. Thus the lines AB and CD cannot intersect at any point P on the right of MN. If we assume that the intersection is at the point P', then a triangle is formed for which the angle 4, congruent to the angle 2, is interior and the angle 6 is exterior not supplementary to it. Then the angle 6 has to be greater than the angle 4, and hence greater than the angle 2, which contradicts the hypothesis. Therefore the lines AB and CD cannot intersect at a point lying on the left of MN either. Thus the lines cannot intersect anywhere, i.e. they are parallel. Similarly, one can prove that $AB \| CD$ if $\angle 1 = \angle 5$, or $\angle 3 = \angle 7$, etc.

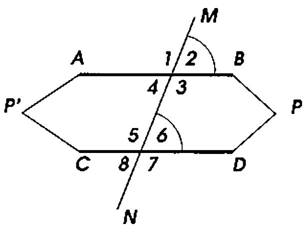

Figure 74

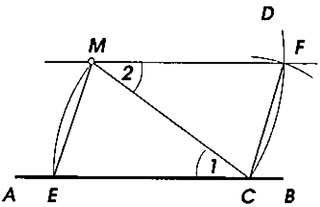

Figure 75

Suppose now that $\angle 4 + \angle 5 = 2d$. Then we conclude that $\angle 4 = \angle 6$ since the sum of angle 6 with the angle 5 is also $2d$. But if $\angle 4 = \angle 6$, then the lines AB and CD cannot intersect, since if they did the angles 4 and 6 (of which one would have been exterior and the other interior not supplementary to it) could not be congruent.

74. Problem. *Through a given point M* (Figure 75), *to construct a line parallel to a given line AB.*

A simple solution to this problem consists of the following. Draw an arc CD of arbitrary radius centered at the point M. Next, draw the arc ME of the same radius centered at the point C. Then draw a small arc of the radius congruent to ME centered at the point C so that it intersects the arc CD at some point F. The line MF will be parallel to AB.

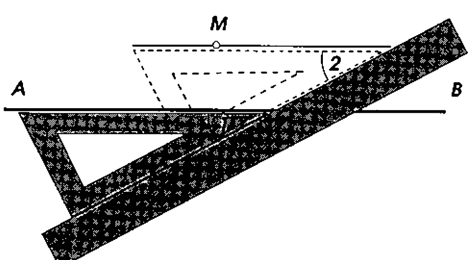

Figure 76

To prove this, draw the auxiliary line MC. The angles 1 and 2 thus formed are congruent by construction (because the triangles EMC and MCF are congruent by the SSS-test), and when alternate angles are congruent, the lines are parallel.

For practical construction of parallel lines it is also convenient to use a drafting triangle and a straightedge as shown in Figure 76.

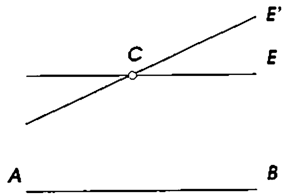

Figure 77

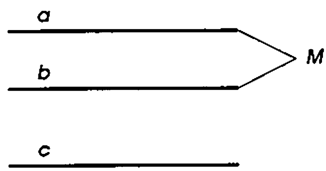

Figure 78

75. The parallel postulate. ***Through a given point, one cannot draw two different lines parallel to the same line.***

Thus, if (Figure 77) $CE \| AB$, then no other line CE' passing through the point C can be parallel to AB, i.e. CE' will meet AB when extended.

It turns out impossible to prove this proposition, i.e. to derive it as a consequence of earlier accepted axioms. It becomes necessary therefore to accept it as a new assumption (postulate, or axiom).

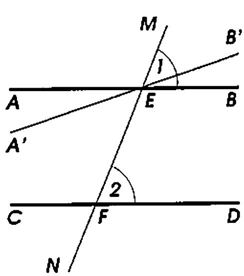

Figure 79

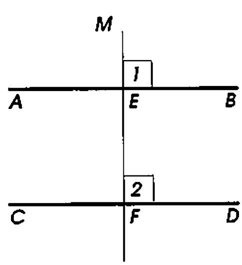

Figure 80

76. `Corollary`. (1) *If* $CE \| AB$ (Figure 77), *and a third line* CE' *intersects one of these two parallel lines, then it intersects the other as well,* because otherwise there would be two different lines CE and CE' passing through the same point C and parallel to AB, which is impossible.

(2) *If each of two lines a and b* (Figure 78) *is parallel to the same third line* c, *then they are parallel to each other.*

Indeed, if we assume that the lines a and b intersect at some point M, there would be two different lines passing through this point and parallel to c, which is impossible.

77. Angles formed by intersection of parallel lines by a transversal.

Theorem (converse to Theorem of §73). ***If two parallel lines*** (AB and CD, Figure 79) ***are intersected by any line*** (MN), ***then:***

(1) ***corresponding angles are congruent;***

(2) ***alternate angles are congruent;***

(3) ***the sum of same-side interior angles is*** $2d$;

(4) ***the sum of same-side exterior angles is*** $2d$.

Let us prove for example that if $AB \| CD$, then the corresponding angles a and b are congruent.

Assume the contrary, i.e. that these angles are not congruent (let us say $\angle 1 > \angle 2$). Constructing $\angle MEB' = \angle 2$ we then obtain a line $A'B'$ distinct from AB and have therefore two lines passing through the point E and parallel to the same line CD. Namely, $AB \| CD$ by the hypothesis of the theorem, and $A'B' \| CD$ due to the congruence of the corresponding angles MEB' and 2. Since this contradicts the parallel postulate, then our assumption that the angles 1 and 2 are not congruent must be rejected; we are left to accept that $\angle 1 = \angle 2$.

Other conclusions of the theorem can be proved the same way.

Corollary. *A perpendicular to one of two parallel lines is perpendicular to the other one as well.*

Indeed, if $AB \| CD$ (Figure 80) and $ME \perp AB$, then firstly ME, which intersects AB, will also intersect CD at some point F, and secondly the corresponding angles 1 and 2 will be congruent. But the angle 1 is right, and thus the angle 2 is also right, i.e. $ME \perp CD$.

78. Tests for non-parallel lines. From the two theorems: direct (§73) and its converse (§75), it follows that the inverse theorems also hold true, i.e.:

If two lines are intersected by a third one in a way such that (1) *corresponding angles are not congruent*, or (2) *alternate interior angles are not congruent*, etc., *then the two lines are not parallel;*

If two lines are not parallel and are intersected by a third one, then (1) *corresponding angles are not congruent,* (2) *alternate interior angles are not congruent,* etc. Among all these tests for non-parallel lines (which are easily proved by *reductio ad absurdum*), the following one deserves special attention:

If the sum of two same-side interior angles (1 and 2, Figure 81) ***differs from*** $2d$, ***then the two lines when extended far enough will intersect,*** since if these lines did not intersect, then they would be parallel, and then the sum of same-side interior angles would be $2d$, which contradicts the hypothesis.

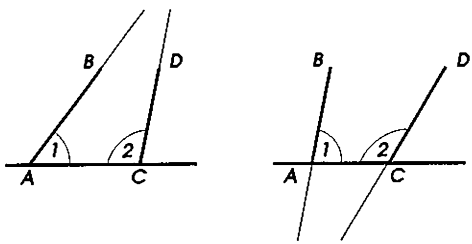

Figure 81

This proposition (supplemented by the statement that the lines intersect on that side of the transversal on which the sum of the same-side interior angles is *smaller than* $2d$) was accepted without proof by the famous Greek geometer **Euclid** (who lived in the 3rd century B.C.) in his ***Elements*** of geometry, and is known as **Euclid's postulate.** Later the preference was given to a simpler formulation: the parallel postulate stated in §75.

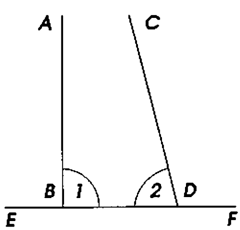

Figure 82

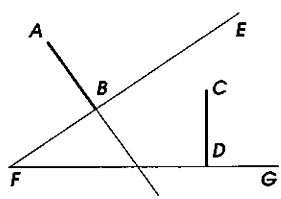

Figure 83

Let us point out two more tests for non-parallelism which will be used later on:

(1) ***A perpendicular*** (AB, Figure 82) ***and a slant*** (CD) ***to the same line*** (EF) ***intersect each other,*** because the sum of same-side interior angles 1 and 2 differs from $2d$.

(2) ***Two lines*** (AB and CD, Figure 83) ***perpendicular to two intersecting lines*** (FE and FG) ***intersect as well.***

Indeed, if we assume the contrary, i.e. that $AB \| CD$, then the line FD, being perpendicular to one of the parallel lines (CD), will be perpendicular to the other (AB), and thus two perpendiculars from the same point F to the same line AB will be dropped, which is impossible.

79. Angles with respectively parallel sides.

Theorem. ***If the sides of one angle are respectively parallel to the sides of another angle, then such angles are either congruent or add up to*** $2d$.

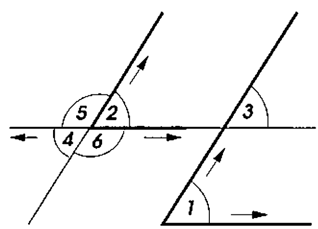

Figure 84

Consider separately the following three cases (Figure 84).

(1) Let the sides of the angle 1 be respectively parallel to the sides of the angle 2 and, beside this, the directions of the respective sides, when counted away from the vertices (as indicated by arrows on the diagram), happen to be the same.

Extending one of the sides of the angle 2 until it meets the non-parallel to it side of the angle 1, we obtain the angle 3 congruent to each of the angles 1 and 2 (as corresponding angles formed by a transversal intersecting parallel lines). Therefore $\angle 1 = \angle 2$.

(2) Let the sides of the angle 1 be respectively parallel to the sides of the angle 2, but the respective sides have opposite directions away from the vertices.

Extending both sides of the angle 4, we obtain the angle 2, which is congruent to the angle 1 (as proved earlier) and to the angle 4 (as vertical to it). Therefore $\angle 4 = \angle 1$.

(3) Finally, let the sides of the angle 1 be respectively parallel to the sides of the angles 5 and 6, and one pair of respective sides have

the same directions, while the other pair, the opposite ones.

Extending one side of the angle 5 or the angle 6, we obtain the angle 2, congruent (as proved earlier) to the angle 1. But $\angle 5$(or $\angle 6$) $+ \angle 2 = 2d$ (by the property of supplementary angles). Therefore $\angle 5$(or $\angle 6$) $+ \angle 1 = 2d$ too.

Thus angles with parallel sides turn out to be congruent when the directions of respective sides away from the vertices are either both the same or both opposite, and when neither condition is satisfied, the angles add up to $2d$.

Remark. One could say that two angles with respectively parallel sides are congruent when both are acute or both are obtuse. In some cases however it is hard to determine *a priori* if the angles are acute or obtuse, so comparing directions of their sides becomes necessary.

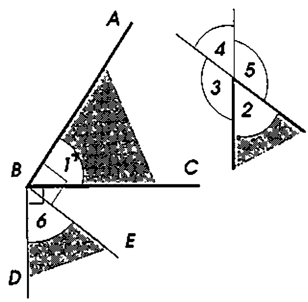

Figure 85

80. Angles with respectively perpendicular sides.

Theorem. ***If the sides of one angle are respectively perpendicular to the sides of another one, then such angles are either congruent or add up to*** $2d$.

Let the angle ABC labeled by the number 1 (Figure 85) be one of the given angles, and the other be one of the four angles 2, 3, 4, 5 formed by two intersecting lines, of which one is perpendicular to the side AB and the other to the side BC.

From the vertex of the angle 1, draw two auxiliary lines: $BD \perp BC$ and $BE \perp BA$. The angle 6 formed by these lines is congruent to the angle 1 for the following reason. The angles DBC and EBA are congruent since both are right. Subtracting from each of them the same angle EBC we obtain: $\angle 1 = \angle 6$. Now notice that the sides of the auxiliary angle 6 are parallel to the intersecting lines which form the angles 2, 3, 4, 5 (because two perpendiculars to the same line are parallel, §71). Therefore the latter angles are either congruent to the angle 6 or supplement it to $2d$. Replacing the angle 6 with the angle 1 congruent to it, we obtain what was required to prove.

EXERCISES

140. Divide the plane by infinite straight lines into five parts, using as few lines as possible.

141. In the interior of a given angle, construct an angle congruent to it.

142. Using a protractor, straightedge, and drafting triangle, measure an angle whose vertex does not fit the page of the diagram.

143. How many axes of symmetry does a pair of parallel lines have? How about three parallel lines?

144. Two parallel lines are intersected by a transversal, and one of the eight angles thus formed is 72°. Find the measures of the remaining seven angles.

145. One of the interior angles formed by a transversal with one of two given parallel lines is $4d/5$. What angle does its bisector make with the other of the two parallel lines?

146. The angle a transversal makes with one of two parallel lines is by 90° greater than with the other. Find the angle.

147. Four out of eight angles formed by a transversal intersecting two given lines contain 60° each, and the remaining four contain 120° each. Does this imply that the given lines are parallel?

148. At the endpoints of the base of a triangle, perpendiculars to the lateral sides are erected. Compute the angle at the vertex of the triangle if these perpendiculars intersect at the angle of 120°.

149. Through a given point, construct a line making a given angle to a given line.

150. Prove that if the bisector of one of the exterior angles of a triangle is parallel to the opposite side, then the triangle is isosceles.

151. In a triangle, through the intersection point of the bisectors of the angles adjacent to a base, a line parallel to the base is drawn. Prove that the segment of this line contained between the lateral sides of the triangle is congruent to the sum of the segments cut out on these sides and adjacent to the base.

*152.** Bisect an angle whose vertex does not fit the page of the diagram.

12 The angle sum of a polygon

81. Theorem. ***The sum of angles of a triangle is*** $2d$.

Let ABC (Figure 86) be any triangle; we are required to prove that the sum of the angles A, B and C is $2d$, i.e. $180°$.

Extending the side AC past C and drawing $CE \| AB$ we find: $\angle A = \angle ECD$ (as corresponding angles formed by a transversal intersecting parallel lines) and $\angle B = \angle BCE$ (as alternate angles formed by a transversal intersecting parallel lines). Therefore

$$\angle A + \angle B + \angle C = \angle ECD + \angle BCE + \angle C = 2d = 180°.$$

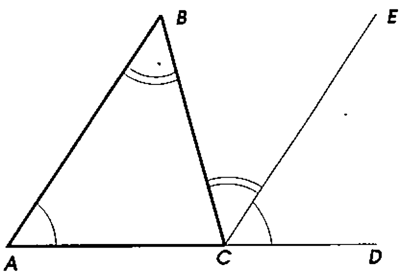

Figure 86

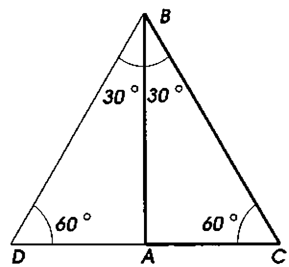

Figure 87

Corollaries. (1) *Any exterior angle of a triangle is congruent to the sum of the interior angles not supplementary to it* (e.g. $\angle BCD = \angle A + \angle B$).

(2) *If two angles of one triangle are congruent respectively to two angles of another, then the remaining angles are congruent as well.*

(3) *The sum of the two acute angles of a right triangle is congruent to one right angle, i.e. it is* $90°$.

(4) *In an isosceles right triangle, each acute angle is* $\frac{1}{2}d$, *i.e.* $45°$.

(5) *In an equilateral triangle, each angle is* $\frac{2}{3}d$, *i.e.* $60°$.

(6) *If in a right triangle* ABC (Figure 87) *one of the acute angles* (for instance, $\angle B$) *is* $30°$, *then the leg opposite to it is congruent to a half of the hypotenuse.* Indeed, noticing that the other acute angle in such a triangle is $60°$, attach to the triangle ABC another triangle ABD congruent to it. Then we obtain the triangle DBC, whose angles are $60°$ each. Such a triangle has to be equilateral (§45), and hence $DC = BC$. But $AC = \frac{1}{2}DC$, and therefore $AC = \frac{1}{2}BC$.

We leave it to the reader to prove the converse proposition: *If a leg is congruent to a half of the hypotenuse, then the acute angle opposite to it is* 30°.

82. Theorem. ***The sum of angles of a convex polygon having*** n ***sides is congruent to two right angles repeated*** $n - 2$ ***times.***

Taking, inside the polygon, an arbitrary point O (Figure 88), connect it with all the vertices. The convex polygon is thus partitioned into as many triangles as it has sides, i.e. n. The sum of angles in each of them is $2d$. Therefore the sum of angles of all the triangles is $2dn$. Obviously, this quantity exceeds the sum of all angles of the polygon by the sum of all those angles which are situated around the point O. But the latter sum is $4d$ (§27). Therefore the sum of angles of the polygon is

$$2dn - 4d = 2d(n-2) = 180^\circ \times (n-2).$$

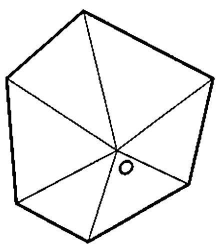

Figure 88

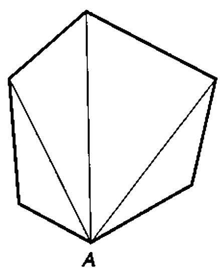

Figure 89

Remarks. (1) The theorem can be also proved this way. From any vertex A (Figure 89) of the convex polygon, draw its diagonals. The polygon is thus partitioned into triangles, the number of which is two less than the number of sides of the polygon. Indeed, if we exclude from counting those two sides which form the angle A of the polygon, then the remaining sides correspond to one triangle each. Therefore the total number of such triangles is $n - 2$, where n denotes the number of sides of the polygon. In each triangle, the sum of angles is $2d$, and hence the sum of angles of all the triangles is $2d(n-2)$. But the latter sum is the sum of all angles of the polygon.

(2) The same result holds true for any non-convex polygon. To prove this, one should first partition it into convex ones. For this, it suffices to extend all sides of the polygon in both directions. The

infinite straight lines thus obtained will divide the plane into convex parts: convex polygons and some infinite regions. The original non-convex polygon will consist of some of these convex parts.

83. Theorem. ***If at each vertex of a convex polygon, we extend one of the sides of this angle, then the sum of the exterior angles thus formed is congruent to*** $4d$ (regardless of the number of sides of the polygon).

Each of such exterior angles (Figure 90) supplements to 2d one of the interior angles of the polygon. Therefore if to the sum of all interior angles we add the sum of these exterior angles, the result will be $2dn$ (where n is the number of sides of the polygon). But the sum of the interior angles, as we have seen, is $2dn - 4d$. Therefore the sum of the exterior angles is the difference:

$$2dn - (2db - 4d) = 2dn - 2dn + 4d = 4d = 360^{\circ}.$$

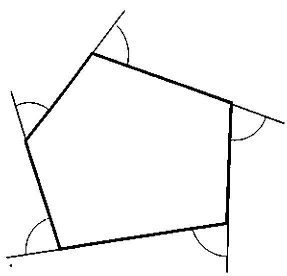

Figure 90

EXERCISES

153. Compute the angle between two medians of an equilateral triangle.

154. Compute the angle between bisectors of acute angles in a right triangle.

155. Given an angle of an isosceles triangle, compute the other two. Consider two cases: the given angle is (a) at the vertex, or (b) at the base.

156. Compute interior and exterior angles of an equiangular pentagon.

157.* Compute angles of a triangle which is divided by one of its bisectors into two isosceles triangles. Find all solutions.

158. Prove that if two angles and the side opposite to the first of them in one triangle are congruent respectively to two angles and the side opposite to the first of them in another triangle, then such triangles are congruent.
`Remark:` This proposition is called sometimes the **AAS-test**, or **SAA-test**.

159. Prove that if a leg and the acute angle opposite to it in one right triangle are congruent respectively to a leg and the acute angle opposite to it in another right triangle, then such triangles are congruent.

160. Prove that in a convex polygon, one of the angles between the bisectors of two consecutive angles is congruent to the semisum of these two angles.

161. Given two angles of a triangle, construct the third one.

162. Given an acute angle of a right triangle, construct the other acute angle.

163. Construct a right triangle, given one of its legs and the acute angle opposite to it.

164. Construct a triangle, given two of its angles and a side opposite to one of them.

165. Construct an isosceles triangle, given its base and the angle at the vertex.

166. Construct an isosceles triangle: (a) given the angle at the base, and the altitude dropped to one of the lateral sides; (b) given the lateral side and the altitude dropped to it.

167. Construct an equilateral triangle, given its altitude.

168. Trisect a right angle (in other words, construct the angle of $\frac{1}{3} \times 90° = 30°$).

169. Construct a polygon congruent to a given one.
`Hint:` Diagonals partition a convex polygon into triangles.

170. Construct a quadrilateral, given three of its angles and the sides containing the fourth angle.
`Hint:` Find the fourth angle.

171.* How many acute angles can a convex polygon have?

172.* Find the sum of the "interior" angles at the five vertices of a five-point star (e.g. the one shown in Figure 221), and the sum of its five exterior angles (formed by extending one of the sides at each vertex). Compare the results with those of §82 and §83.

173.* Following Remark (2) in §82, extend the results of §82 and §83 to non-convex polygons.

13 Parallelograms and trapezoids

84. The parallelogram. A quadrilateral whose opposite sides are pairwise parallel is called a **parallelogram**. Such a quadrilateral ($ABCD$, Figure 91) is obtained, for instance, by intersecting any two parallel lines KL and MN with two other parallel lines RS and PQ.

85. Properties of sides and angles.

Theorem. ***In any parallelogram, opposite sides are congruent, opposite angles are congruent, and the sum of angles adjacent to one side is*** $2d$ (Figure 92).

Drawing the diagonal BD we obtain two triangles: ABD and BCD, which are congruent by the ASA-test because BD is their common side, $\angle 1 = \angle 4$, and $\angle 2 = \angle 3$ (as alternate angles formed by a transversal intersecting parallel lines). It follows from the congruence of the triangles that $AB = CD$, $AD = BC$, and $\angle A = \angle C$. The opposite angles B and D are also congruent since they are sums of congruent angles.

Finally, the angles adjacent to one side, e.g. the angles A and D, add up to $2d$ since they are same-side interior angles formed by a transversal intersecting parallel lines.

Corollary. *If one of the angles of a parallelogram is right, then the other three are also right.*

Remark. The congruence of the opposite sides of a parallelogram can be rephrased this way: *parallel segments cut out by parallel lines are congruent.*

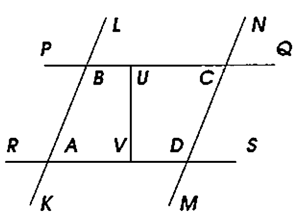

Figure 91

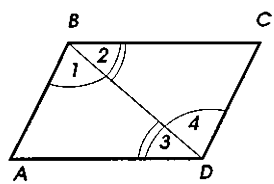

Figure 92

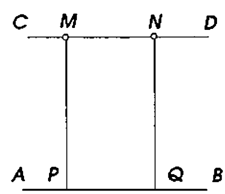

Figure 93

Corollary. *If two lines are parallel, then all points of each of them are the same distance away from the other line;* in short *parallel lines* (AB and CD, Figure 93) *are everywhere the same distance apart.*

Indeed, if from any two points M and N of the line CD, the perpendiculars MP and NQ to AB are dropped, then these perpendiculars are parallel (§71), and therefore the quadrilateral $MNQP$ is a parallelogram. It follows that $MN = NQ$, i.e. the points M and N are are the same distance away from the line AB.

Remark. Given a parallelogram ($ABCD$, Figure 91), one sometimes refers to a pair of its parallel sides (e.g. AD and BC) as a pair of **bases**. In this case, a line segment (UV) connecting the parallel lines PQ and RS and perpendicular to them is called an **altitude** of the parallelogram. Thus, the corollary can be rephrased this way: *all altitudes between the same bases of a parallelogram are congruent to each other.*

86. Two tests for parallelograms.

Theorem. ***If in a convex quadrilateral:***

(1) ***opposite sides are congruent to each other,*** or

(2) ***two opposite sides are congruent and parallel,***

then this quadrilateral is a parallelogram.

(1) Let $ABCD$ (Figure 92) be a quadrilateral such that

$$AB = CD \quad \text{and} \quad BC = AD.$$

It is required to prove that this quadrilateral is a parallelogram, i.e. that $AB \| CD$ and $BC \| AD$.

Drawing the diagonal BD we obtain two triangles, which are congruent by the SSS-test since BD is their common side, and $AB = CD$ and $BC = AD$ by hypothesis. It follows from the congruence of the triangles that $\angle 1 = \angle 4$ and $\angle 2 = \angle 3$ (in congruent triangles, congruent sides oppose congruent angles). This implies that $AB \| CD$ and $BC \| AD$ (if alternate angles are congruent, then the lines are parallel).

(2) Let $ABCD$ (Figure 92) be a quadrilateral such that $BC \| AD$ and $BC = AD$. It is required to prove that $ABCD$ is a parallelogram, i.e. that $AB \| CD$.

The triangles ABD and BCD are congruent by the SAS-test because BD is their common side, $BC = AD$ (by hypothesis), and $\angle 2 = \angle 3$ (as alternate angles formed by intersecting parallel lines by a transversal). The congruence of the triangles implies that $\angle 1 = \angle 4$, and therefore $AB \| CD$.

87. The diagonals and their property.

Theorem. (1) ***If a quadrilateral*** ($ABCD$, Figure 94) ***is a parallelogram, then its diagonals bisect each other.***

(2) *Vice versa,* ***in a quadrilateral, if the diagonals bisect each other, then this quadrilateral is a parallelogram.***

(1) The triangles BOC and AOD are congruent by the ASA-test, because $BC = AD$ (as opposite sides of a parallelogram), $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (as alternate angles). It follows from the congruence of the triangles that $OA = OC$ and $OD = OB$.

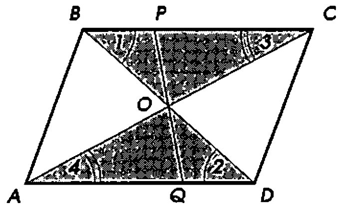

Figure 94

(2) If $AO = OC$ and $BO = OD$, then the triangles AOD and BOC are congruent (by the SAS-test). It follows from the congruence of the triangles that $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Therefore $BC \| AD$ (alternate angles are congruent) and $BC = AD$. Thus $ABCD$ is a parallelogram (by the second test).

88. Central symmetry. Two points A and A' (Figure 95) are called **symmetric** about a point O, if O is the midpoint of the line segment AA'.

Thus, in order to construct the point symmetric to a given point A about another given point O, one should connect the points A and O by a line, extend this line past the point O, and mark on the extension the segment OA' congruent to OA. Then A' is the required point.

Two figures (or two parts of the same figure) are called symmetric about a given point O, if for each point of one figure, the point symmetric to it about the point O belongs to the other figure, and *vice versa*. The point O is then called the **center of symmetry**. The symmetry itself is called *central* (as opposed to the *axial* symmetry we encountered in §37). If each point of a figure is symmetric to some point of the same figure (about a certain center), then the figure is said to have a center of symmetry. An example of such a figure is a circle; its center of symmetry is the center of the circle.

Every figure can be superimposed on the figure symmetric to it by rotating the figure through the angle* 180° *about the center of symmetry. Indeed, any two symmetric points (say, A and A', Figure 95) exchange their positions under this rotation.

Remarks. (1) Two figures symmetric about a point can be superimposed therefore by a motion *within* the plane, i.e. *without* lifting them off the plane. In this regard central symmetry differs from axial symmetry (§37), where for superimposing the figures it was necessary to flip one of them over.

(2) Just like axial symmetry, central symmetry is frequently found around us (see Figure 96, which indicates that each of the letters N and S has a center of symmetry while E and W do not).

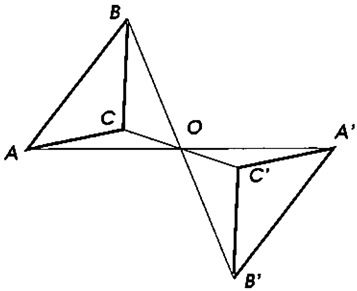

Figure 95

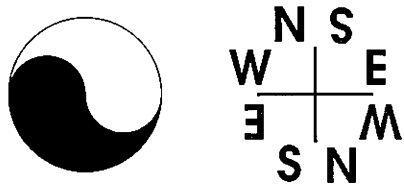

Figure 96

89. ***In a parallelogram, the intersection point of the diagonals is the center of symmetry*** (Figure 94).

Indeed, the vertices A and C are symmetric about the intersection point O of the diagonals (since $AO = OD$), and so are B and C. Furthermore, for a point P on the boundary of the parallelogram, draw the line PO, and let Q be the point where the extension of line past O meets the boundary. The triangles AQO and CPO are congruent by the ASA-test for $\angle 4 = \angle 3$ (as alternate), $\angle QOA = \angle POC$ (as vertical), and $AO = OC$. Therefore $QO = OP$, i.e. the points P and Q are symmetric about the center O.

Remark. If a parallelogram is turned around 180° about the intersection point of the diagonals, then each vertex exchanges its position with the opposite one (A with C, and B with D in Figure 94), and the new position of the parallelogram will coincide with the old one.

Most parallelograms do not possess axial symmetry. In the next section we will find out which of them do.

90. The rectangle and its properties. If one of the angles of a parallelogram is right then the other three are also right (§85). A parallelogram all of whose angles are right is called a **rectangle**.

Since rectangles are parallelograms, they possess all properties of parallelograms (for instance, their diagonals bisect each other, and the intersection point of the diagonals is the center of symmetry). However rectangles have their own special properties.

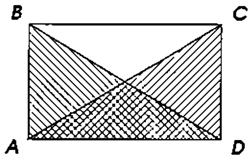

Figure 97

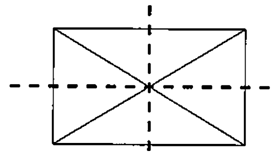

Figure 98

(1) ***In a rectangle*** ($ABCD$, Figure 97), ***the diagonals are congruent.***

The right triangles ACD and ABD are congruent because they have respectively congruent legs (AD is a common leg, and $AB = CD$ as opposite sides of a parallelogram). The congruence of the triangles implies: $AC = BD$.

(2) ***A rectangle has two axes of symmetry.*** Namely, each line passing through the center of symmetry and parallel to two opposite sides of the rectangle is its axis of symmetry. The axes of symmetry of a rectangle are perpendicular to each other (Figure 98).

91. The rhombus and its properties. A parallelogram all of whose sides are congruent is called a **rhombus**. Beside all the properties that parallelograms have, rhombi also have the following special ones.

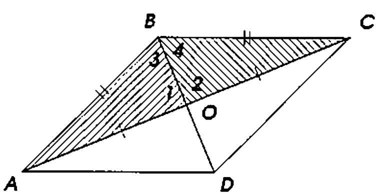

Figure 99

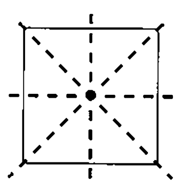

Figure 100

(1) ***Diagonals of a rhombus*** ($ABCD$, Figure 99) ***are perpendicular and bisect the angles of the rhombus.***

The triangles AOB and COB are congruent by the SSS-test because BO is their common side, $AB = BC$ (since all sides of a rhombus are congruent), and $AO = OC$ (since the diagonals of any parallelogram bisect each other). The congruence of the triangles implies that

$$\angle 1 = \angle 2, \quad \text{i.e. } BD \perp AC, \quad \text{and} \quad \angle 3 = \angle 4,$$

i.e. the angle B is bisected by the diagonal BD. From the congruence of the triangles BOC and DOC, we conclude that the angle C is bisected by the diagonal CA, etc.

(2) ***Each diagonal of a rhombus is its axis of symmetry.*** The diagonal BD (Figure 99) is an axis of symmetry of the rhombus $ABCD$ because by rotating $\triangle BAD$ about BD we can superimpose it onto $\triangle BCD$. Indeed, the diagonal BD bisects the angles B and D, and beside this $AB = BC$ and $AD = DC$.

The same reasoning applies to the diagonal AC.

92. The square and its properties. A **square** can be defined as a parallelogram all of whose sides are congruent and all of whose angles are right. One can also say that a square is a rectangle all of whose sides are congruent, or a rhombus all of whose angles are right. Therefore a square possesses all the properties of parallelograms, rectangles and rhombi. For instance, a square has four axes of symmetry (Figure 100): two passing through the midpoints of opposite sides (as in a rectangle), and two passing through the vertices of the opposite angles (as in a rhombus).

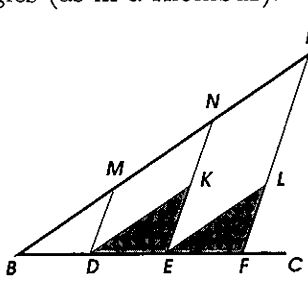

Figure 101

93. A theorem based on properties of parallelograms.

Theorem. *If on one side of an angle* (e.g. on the side BC of the angle ABC, Figure 101), *we mark segments congruent to each other* ($DE = EF = \ldots$), *and through their endpoints, we draw parallel lines* (DM, EN, FP, $\ldots$) *until their intersections with the other side of the angle, then the segments cut out on this side will be congruent to each other* ($MN = NP = \ldots$).

Draw the auxiliary lines DK and DL parallel to AB. The triangles DKE and ELF are congruent by the ASA-test since $DE = EF$ (by hypothesis), and $\angle KDE = \angle LEF$ and $\angle KED = \angle LFE$ (as corresponding angles formed by a transversal intersecting parallel lines). From the congruence of the triangles, it follows that $DK = EL$. But $DK = MN$ and $EL = NP$ (as opposite sides of parallelograms), and therefore $MN = NP$.

Remark. The congruent segments can be also marked starting from the vertex of the angle B, i.e. like this: $BD = DE = EF = \dots$. Then the congruent segments on the other side of the angle are also formed starting from the vertex, i.e. $BM = MN = NP = \dots$.

94. Corollary. *The line* (DE, Figure 102) *passing through the midpoint of one side* (AB) *of a triangle and parallel to another side bisects the third side* (BC).

Indeed, on the side of the angle B, two congruent segments $BD = DA$ are marked and through the division points D and A, two parallel lines DE and AC are drawn until their intersections with the side BC. Therefore, by the theorem, the segments cut out on this side are also congruent, i.e. $BE = EC$, and thus the point E bisects BC.

Remark. The segment connecting the midpoints of two sides of a triangle is called a **midline** of this triangle.

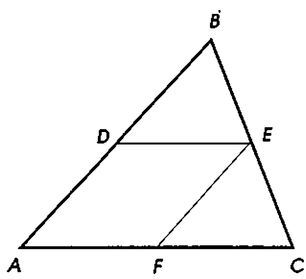

Figure 102

95. The midline theorem.

Theorem. ***The line segment*** (DE, Figure 102) ***connecting the midpoints of two sides of a triangle is parallel to the third side, and is congruent to a half of it.***

To prove this, imagine that through the midpoint D of the side AB, we draw a line parallel to the side AC. Then by the result of §94, this line bisects the side BC and thus coincides with the line DE connecting the midpoints of the sides AB and BC.

Furthermore, drawing the line $EF \| AD$, we find that the side

AC is bisected at the point F. Therefore $AF = FC$ and beside this $AF = DE$ (as opposite sides of the parallelogram $ADEF$). This implies: $DE = \frac{1}{2}AC$.

96. The trapezoid. A quadrilateral which has two opposite sides parallel and the other two opposite sides non-parallel is called a **trapezoid**. The parallel sides (AD and BC, Figure 103) of a trapezoid are called its **bases**, and the non-parallel sides (AB and CD) its **lateral sides**. If the lateral sides are congruent, the trapezoid is called **isosceles**.

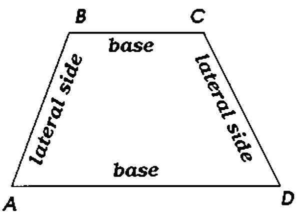

Figure 103

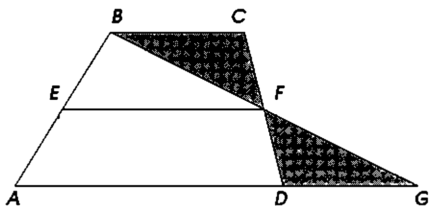

Figure 104

97. The midline of a trapezoid. The line segment connecting the midpoints of the lateral sides of a trapezoid is called its **midline**.

Theorem. ***The midline*** (EF, Figure 104) ***of a trapezoid is parallel to the bases and is congruent to their semisum.***

Through the points B and F, draw a line until its intersection with the extension of the side AD at some point G. We obtain two triangles: BCF and GDF, which are congruent by the ASA-test since $CF = FD$ (by hypothesis), $\angle BFC = \angle GFD$ (as vertical angles), and $\angle BCF = \angle GDF$ (as alternate interior angles formed by a transversal intersecting parallel lines). From the congruence of the triangles, it follows that $BF = FG$ and $BC = DG$. We see now that in the triangle ABG, the line segment EF connects the midpoints of two sides. Therefore (§95) we have: $EF \| AG$ and $EF = \frac{1}{2}(AD+DG)$, or in other words, $EF \| AD$ and $EF = \frac{1}{2}(AD + BC)$.

EXERCISES

174. Is a parallelogram considered a trapezoid?

175. How many centers of symmetry can a polygon have?

176. Can a polygon have two parallel axes of symmetry?

177. How many axes of symmetry can a quadrilateral have?

Prove theorems:

178. Midpoints of the sides of a quadrilateral are the vertices of a parallelogram. Determine under what conditions this parallelogram will be (a) a rectangle, (b) a rhombus, (c) a square.

179. In a right triangle, the median to the hypotenuse is congruent to a half of it.

Hint: Double the median by extending it past the hypotenuse.

180. Conversely, if a median is congruent to a half of the side it bisects, then the triangle is right.

181. In a right triangle, the median and the altitude drawn to the hypotenuse make an angle congruent to the difference of the acute angles of the triangle.

182. In $\triangle ABC$, the bisector of the angle A meets the side BC at the point D; the line drawn from D and parallel to CA meets AB at the point E; the line drawn from E and parallel to BC meets AC at F. Prove that $EA = FC$.

183. Inside a given angle, another angle is constructed such that its sides are parallel to the sides of the given one and are the same distance away from them. Prove that the bisector of the constructed angle lies on the bisector of the given angle.

184. The line segment connecting any point on one base of a trapezoid with any point on the other base is bisected by the midline of the trapezoid.

185. The segment between midpoints of the diagonals of a trapezoid is congruent to the semidifference of the bases.

186. Through the vertices of a triangle, the lines parallel to the opposite sides are drawn. Prove that the triangle formed by these lines consists of four triangles congruent to the given one, and that each of its sides is twice the corresponding side of the given triangle.

187. In an isosceles triangle, the sum of the distances from each point of the base to the lateral sides is constant, namely it is congruent to the altitude dropped to a lateral side.

188. How does this theorem change if points on the extension of the base are taken instead?

189. In an equilateral triangle, the sum of the distances from an interior point to the sides of this triangle does not depend on the point, and is congruent to the altitude of the triangle.

190. A parallelogram whose diagonals are congruent is a rectangle.

191. A parallelogram whose diagonals are perpendicular to each other is a rhombus.

192. Any parallelogram whose angle is bisected by the diagonal is a rhombus.

193. From the intersection point of the diagonals of a rhombus, perpendiculars are dropped to the sides of the rhombus. Prove that the feet of these perpendiculars are vertices of a rectangle.

194. Bisectors of the angles of a rectangle cut out a square.

195. Let A', B', C', and D' be the midpoints of the sides CD, DA, AB, and BC of a square. Prove that the segments AA', CC', DD', and BB' cut out a square, whose sides are congruent to 2/5th of any of the segments.

196. Given a square $ABCD$. On its sides, congruent segments AA', BB', CC', and DD' are marked. The points A', B', C', and D' are connected consecutively by lines. Prove that $A'B'C'D'$ is a square.

Find the geometric locus of:

197. The midpoints of all segments drawn from a given point to various points of a given line.

198. The points equidistant from two given parallel lines.

199. The vertices of triangles having a common base and congruent altitudes.

Construction problems

200. Draw a line parallel to a given one and situated at a given distance from it.

201. Through a given point, draw a line such that its line segment, contained between two given lines, is bisected by the given point.

202. Through a given point, draw a line such that its line segment, contained between two given parallel lines, is congruent to a given segment.

203. Between the sides of a given angle, place a segment congruent to a given segment and perpendicular to one of the sides of the angle.

204. Between the sides of a given angle, place a segment congruent to a given segment and parallel to a given line intersecting the sides of the angle.

205. Between the sides of a given angle, place a segment congruent to a given segment and such that it cuts congruent segments on the sides of the angle.

206. In a triangle, draw a line parallel to its base and such that the line segment contained between the lateral sides is congruent to the sum of the segments cut out on the lateral sides and adjacent to the base.

14 Methods of construction and symmetries

98. Problem. *To divide a given line segment* (AB, Figure 105) *into a given number of congruent parts* (e.g. into 3).

From the endpoint A, draw a line AC that forms with AB some angle. Mark on AC, starting from the point A, three congruent segments of arbitrary length: $AD = DE = EF$. Connect the point F with B, and draw through E and D lines EN and DM parallel to FB. Then, by the results of §93, the segment AB is divided by the points M and N into three congruent parts.

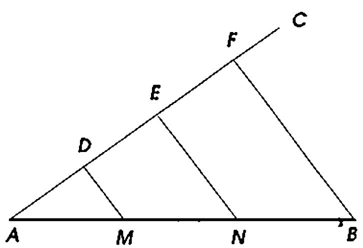

Figure 105

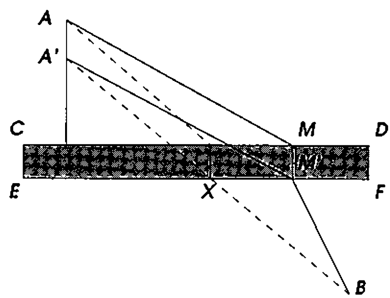

Figure 106

99. The method of parallel translation. A special method of solving construction problems, known as the method of parallel translation, is based on properties of parallelograms. It can be best explained with an example.

Problem. *Two towns A and B* (Figure 106) *are situated on opposite sides of a canal whose banks CD and EF are parallel straight lines. At which point should one build a bridge MM' across the canal in order to make the path $AM + MM' + M'B$ between the towns the shortest possible?*

To facilitate the solution, imagine that all points of the side of the canal where the town A is situated are moved downward ("translated") the same distance along the lines perpendicular to the banks of the canal as far as to make the bank CD merge with the bank EF. In particular, the point A is translated to the new position A' on the perpendicular AA' to the banks, and the segment AA' is congruent to the bridge MM'. Therefore $AA'M'M$ is a parallelogram (§86 (2)), and hence $AM = A'M'$. We conclude that the sum $AM + MM' + M'B$ is congruent to $AA' + A'M' + M'B$. The latter sum will be the shortest when the broken line $A'M'B$ is straight.

Thus the bridge should be built at that point X on bank EF where the bank intersects with the straight line $A'B$.

100. The method of reflection. Properties of axial symmetry can also be used in solving construction problems. Sometimes the required construction procedure is easily discovered when one folds a part of the diagram along a certain line (or, equivalently, **reflects** it in this line as in a mirror) so that this part occupies the symmetric position on the other side of the line. Let us give an example.

Problem. *Two towns A and B* (Figure 107) *are situated on the same side of a railroad CD which has the shape of a straight line. At which point on the railroad should one build a station M in order to make the sum $AM + MB$ of the distances from the towns to the station the smallest possible?*

Reflect the point A to the new position A' symmetric about the line CD. The segment $A'M$ is symmetric to AM about the line CD, and therefore $A'M = AM$. We conclude that the sum $AM + MB$ is congruent to $A'M + MB$. The latter sum will be the smallest when the broken line $A'MB$ is straight. Thus the station should be built at the point X where the railroad line CD intersects the straight line $A'B$.

The same construction solves yet another problem: given the line CD, and the points A and B, find a point M such that $\angle AMC = \angle BMD$.

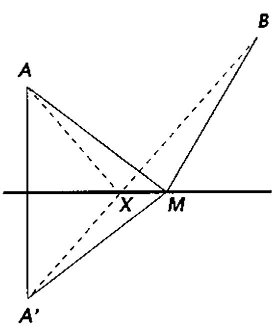

Figure 107

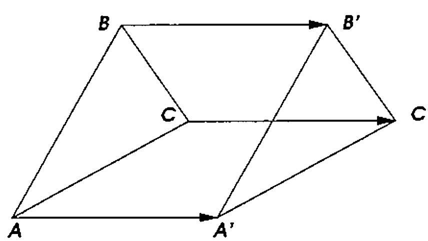

Figure 108

101. Translation. Suppose that a figure (say, a triangle ABC, Figure 108) is moved to a new position ($A'B'C'$) in a way such that all segments between the points of the figure remain parallel to themselves (i.e. $A'B' \| AB$, $B'C' \| BC$, etc.). Then the new figure is called a **translation** of the original one, and the whole motion, too, is called translation. Thus the sliding motion of a drafting triangle (Figure 76) along a straightedge (in the construction of parallel lines described in §74) is an example of translation.

Note that by the results of §86, if $AB \| A'B'$ and $AB = A'B'$ (Figure 108), then $ABB'A'$ is a parallelogram, and therefore $AA' \| BB'$ and $AA' = BB'$. Thus, if under translation of a figure, the new position A' of one point A is known, then in order to translate all other points B, C, etc., it suffices to construct the parallelograms $AA'B'B$, $AA'C'C$, etc. In other words, it suffices to construct line segments BB', CC', etc. parallel to the line segment AA', directed the same way as AA', and congruent to it.

Vice versa, if we move a figure (e.g. $\triangle ABC$) to a new position ($\triangle A'B'C'$) by constructing the line segments AA', BB', CC', etc. which are congruent and parallel to each other, and are also directed the same way, then the new figure is a translation of the old one. Indeed, the quadrilaterals $AA'B'B$, $AA'C'C$, etc. are parallelograms, and therefore all the segments AB, BC, etc. are moved to their new positions $A'B'$, $B'C'$, etc. remaining parallel to themselves.

Let us give one more example of a construction problem solved by the method of translation.

102. `Problem.` *To construct a quadrilateral ABCD (Figure 109), given segments congruent to its sides and to the line EF connecting the midpoints of two opposite sides.*

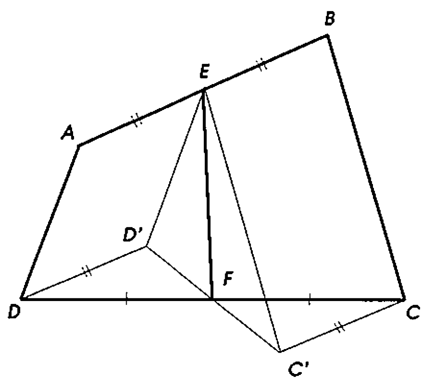

Figure 109

To bring the given lines close to each other, translate the sides AD and BC, i.e. move them in a way such that they remain parallel to themselves, to the new positions ED' and EC'. Then $DAED'$ and $C'EBC$ are parallelograms, and hence the segment DD' is congruent and parallel to AE, and the segment CC' congruent and parallel to BE. But $AE = EB$, and therefore $DD' = CC'$ and $DD' \| CC'$. As a consequence, the triangles $DD'F$ and $CC'F$ are congruent by the SAS-test (since $DD' = CC'$, $DF = FC$, and $\angle D'DF = \angle C'CF$). The congruence of the triangles implies that $\angle D'FD = \angle C'FC$, hence the broken line $D'FC'$ turns out to be straight, and therefore the figure $ED'FC'$ is a triangle. In this triangle, two sides are known ($ED' = AD$ and $EC' = BC$), and the median EF to the third side is known too. The triangle $EC'D'$ is easily recovered from these data. (Namely, double EF by extending it past F and connect the obtained endpoint with D' and C'. In the resulting parallelogram, all sides and one of the diagonals are known.)

Having recovered $\triangle ED'C'$, construct the triangles $D'DF$ and $C'CF$, and then the entire quadrilateral $ABCD$.

EXERCISES

207. Construct a triangle, given:
(a) its base, the altitude, and a lateral side;
(b) its base, the altitude, and an angle at the base;
(c) an angle, and two altitudes dropped to the sides of this angle;
(d) a side, the sum of the other two sides, and the altitude dropped to one of these sides;
(e) an angle at the base, the altitude, and the perimeter.

208. Construct a quadrilateral, given three of its sides and both diagonals.

209. Construct a parallelogram, given:
(a) two non-congruent sides and a diagonal;
(b) one side and both diagonals;
(c) the diagonals and the angle between them;
(d) a side, the altitude, and a diagonal. (Is this always possible?)

210. Construct a rectangle, given a diagonal and the angle between the diagonals.

211. Construct a rhombus, given:
(a) its side and a diagonal;
(b) both diagonals;
(c) the distance between two parallel sides, and a diagonal;
(d) an angle, and the diagonal passing through its vertex;
(e) a diagonal, and an angle opposite to it;
(f) a diagonal, and the angle it forms with one of the sides.

212. Construct a square, given its diagonal.

213. Construct a trapezoid, given:
(a) its base, an angle adjacent to it, and both lateral sides (there can be two solutions, one, or none);
(b) the difference between the bases, a diagonal, and lateral sides;
(c) the four sides (is this always possible?);
(d) a base, its distance from the other base, and both diagonals (when is this possible?);
(e) both bases and both diagonals (when is this possible?).

*214.** Construct a square, given:
(a) the sum of a diagonal and a side;
(b) the difference of a diagonal and an altitude.

*215.** Construct a parallelogram, given its diagonals and an altitude.

*216.** Construct a parallelogram, given its side, the sum of the diagonals, and the angle between them.

*217.** Construct a triangle, given:
(a) two of its sides and the median bisecting the third one;
(b) its base, the altitude, and the median bisecting a lateral side.

*218.** Construct a right triangle, given:
(a) its hypotenuse and the sum of the legs;
(b) the hypotenuse and the difference of the legs. Perform the research stage of the solutions.

219. Given an angle and a point inside it, construct a triangle with the shortest perimeter such that one of its vertices is the given point and the other two vertices lie on the sides of the angle.
`Hint:` use the method of reflection.

*220.** Construct a quadrilateral $ABCD$ whose sides are given assuming that the diagonal AC bisects the angle A.

*221.** Given positions A and B of two billiard balls in a rectangular billiard table, in what direction should one shoot the ball A so that it reflects consecutively in the four sides of the billiard and then hits the ball B?

222. Construct a trapezoid, given all of its sides.
`Hint:` use the method of translation.

*223.** Construct a trapezoid, given one of its angles, both diagonals, and the midline.

*224.** Construct a quadrilateral, given three of its sides and both angles adjacent to the unknown side.

Chapter 2

THE CIRCLE

1 Circles and chords

103. Preliminary remarks. Obviously, through a point (A, Figure 110), it is possible to draw as many circles as one wishes: their centers can be chosen arbitrarily. Through two points (A and B, Figure 111), it is also possible to draw unlimited number of circles, but their centers cannot be arbitrary since the points equidistant from two points A and B must lie on the **perpendicular bisector** of the segment AB (i.e. on the perpendicular to the segment AB passing through its midpoint, §56).

Let us find out if it is possible to draw a circle through three points.

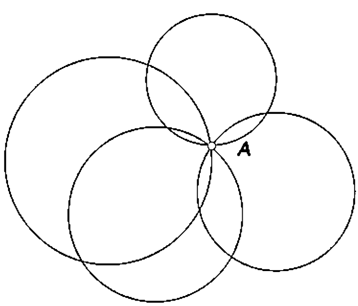

Figure 110

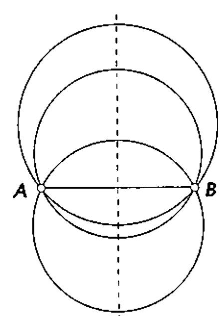

Figure 111

104. Theorem. *Through any three points, not lying on the same line, it is possible to draw a circle, and such a circle is unique.*

Through three points A, B, C (Figure 112), not lying on the same line, (in other words, through the vertices of a triangle ABC), it is possible to draw a circle only if there exists a fourth point O, which is equidistant from the points A, B, and C. Let us prove that such a point exists and is unique. For this, we take into account that any point equidistant from the points A and B must lie on the perpendicular bisector MN of the side AB (§56). Similarly, any point equidistant from the points B and C must lie on the perpendicular bisector PQ of the side BC. Therefore, if a point equidistant from the three points A, B, and C exists, it must lie on both MN and PQ, which is possible only when it coincides with the intersection point of these two lines. The lines MN and PQ do intersect (since they are perpendicular to the intersecting lines AB and BC, §78). The intersection point O will be equidistant from A, B, and C. Thus, if we take this point for the center, and take the segment OA (or OB, or OC) for the radius, then the circle will pass through the points A, B, and C. Since the lines MN and PQ can intersect only at one point, the center of such a circle is unique. The length of the radius is also unambiguous, and therefore the circle in question is unique.

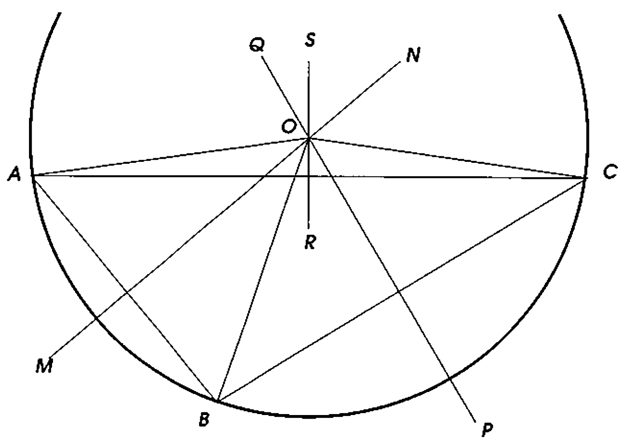

Figure 112

Remarks. (1) If the points A, B, and C (Figure 112) lay on the same line, then the perpendiculars MN and PQ would have been parallel, and therefore could not intersect. Thus, through three points lying on the same line, it is not possible to draw a circle.

(2) Three or more points lying on the same line are often called **collinear**.

Corollary. The point O, being the same distance away from A and C, has to also lie on the perpendicular bisector RS of the side AC. Thus: *three perpendicular bisectors of the sides of a triangle intersect at one point.*

105. Theorem. ***The diameter*** (AB, Figure 113), ***perpendicular to a chord, bisects the chord and each of the two arcs subtended by it.***

Fold the diagram along the diameter AB so that the left part of the diagram falls onto the right one. Then the left semicircle will be identified with the right semicircle, and the perpendicular KC will merge with KD. It follows that the point C, which is the intersection of the semicircle and KC, will merge with D. Therefore $KC = KD$, $\overset{\frown}{BC} = \overset{\frown}{BD}$, $\overset{\frown}{AC} = \overset{\frown}{AD}$.

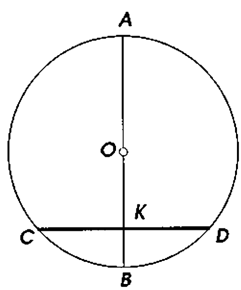

Figure 113

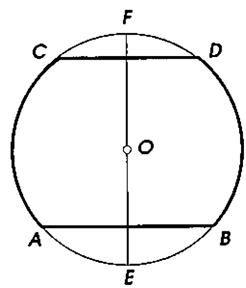

Figure 114

106. Converse theorems. (1) ***The diameter*** (AB), ***bisecting a chord*** (CD), ***is perpendicular to this chord and bisects the arc subtended by it*** (Figure 113).

(2) ***The diameter*** (AB), ***bisecting an arc*** (CBD), ***is perpendicular to the chord subtending the arc, and bisects it.***

Both propositions are easily proved by *reductio ad absurdum.*

107. Theorem. ***The arcs*** (AC and BD, Figure 114) ***contained between parallel chords*** (AB and CD) ***are congruent.***

Fold the diagram along the diameter $EF \perp AB$. Then we can conclude on the basis of the previous theorem that the point A merges with B, and the point C with D. Therefore the arc AC is identified with the arc BD, i.e. these arcs are congruent.

108. Problems. (1) *To bisect a given arc* (AB, Figure 115).

Connecting the ends of the arc by the chord AB, drop the perpendicular to this chord from the center and extend it up to the intersection point with the arc. By the result of §106, the arc AB is bisected by this perpendicular.

However, if the center is unknown, then one should erect the perpendicular to the chord at its midpoint.

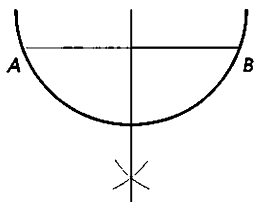

Figure 115

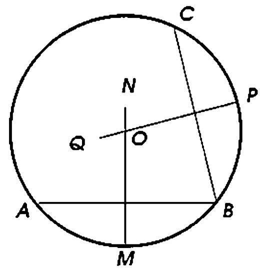

Figure 116

(2) *To find the center of a given circle* (Figure 116).

Pick on the circle any three points A, B, and C, and draw two chords through them, for instance, AB and BC. Erect perpendiculars MN and PQ to these chords at their midpoints. The required center, being equidistant from A, B, and C, has to lie on MN and PQ. Therefore it is located at the intersection point O of these perpendiculars.

109. Relationships between arcs and chords.

Theorems. ***In a disk, or in congruent disks:***

(1) ***if two arcs are congruent, then the chords subtending them are congruent and equidistant from the center;***

(2) ***if two arcs, which are smaller than the semicircle, are not congruent, then the greater of them is subtended by the greater chord, and the greater of the two chords is closer to the center.***

(1) Let an arc AB (Figure 117) be congruent to the arc CD; it is required to prove that the chords AB and CD are congruent, and that the perpendiculars OE and OF to the chords dropped from the center are congruent too.

Rotate the sector AOB about the center O so that the radius OA coincides with the radius OC. Then the arc AB will go along the arc CD, and since the arcs are congruent they will coincide. Therefore the chord AB will coincide with the chord CD, and the perpendicular OE will merge with OF (since the perpendicular from a given point to a given line is unique), i.e. $AB = CD$ and $OE = OF$.

(2) Let the arc AB (Figure 118) be smaller than the arc CD, and let both arcs be smaller than the semicircle; it is required to prove that the chord AB is smaller than the chord CD, and that the perpendicular OE is greater than the perpendicular OF.

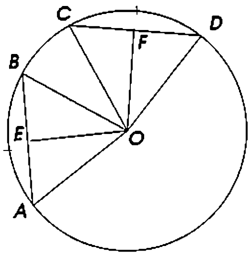

Figure 117

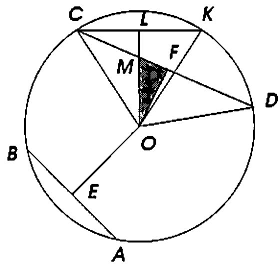

Figure 118

Mark on the arc CD the arc CK congruent to the arc AB and draw the auxiliary chord CK, which by the result of part (1) is congruent to and is the same distance away from the center as the chord AB. The triangles COD and COK have two pairs of respectively congruent sides (since they are radii), and the angles contained between these sides are not congruent. In this case (§50), the greater angle (i.e. $\angle COD$) is opposed by the greater side. Thus $CD > CK$, and therefore $CD > AB$.

In order to prove that $OE > OF$, draw $OL \perp CK$ and take into account that $OE = OL$ by the result of part (1), and therefore it suffices to compare OF with OL. In the right triangle OFM (shaded in Figure 118), the hypotenuse OM is greater than the leg OF. But $OL > OM$, and hence $OL > OF$, i.e. $OE > OF$.

The theorem just proved for *one* disk remains true for *congruent* disks because such disks differ from one another only by their position.

110. Converse theorems. Since the previous theorems address all possible mutually exclusive cases of comparative size of two arcs of the same radius (assuming that the arcs are smaller than the semicircle), and the obtained conclusions about comparative size of subtending chords or their distances from the center are mutually exclusive too, the converse propositions have to hold true as well. Namely:

In a disk, or in congruent disks:

(1) ***congruent chords are equidistant from the center and subtend congruent arcs;***

(2) ***chords equidistant from the center are congruent and subtend congruent arcs***;

(3) ***the greater one of two non-congruent chords is closer to the center and subtends the greater arc***;

(4) ***among two chords non-equidistant to the center, the one which is closer to the center subtends the greater arc.***

These propositions are easy to prove by *reductio ad absurdum.* For instance, to prove the first of them we may argue this way. If the given chords subtended non-congruent arcs, then due to the first direct theorem the chords would have been non-congruent, which contradicts the hypothesis. Therefore congruent chords must subtend congruent arcs. But when the arcs are congruent, then by the direct theorem, the subtending chords are equidistant from the center.

111. Theorem. ***A diameter is the greatest of all chords.***

Connecting the center O with the ends of any chord AB not passing through the center (Figure 119), we obtain a triangle AOB such that the chord AB is one of its sides, and the other two sides are radii. By the triangle inequality (§48) we conclude that the chord AB is smaller than the sum of two radii, while a diameter is the sum of two radii. Thus a diameter is greater than any chord not passing through the center. But since a diameter is also a chord, one can say that diameters are the greatest of all chords.

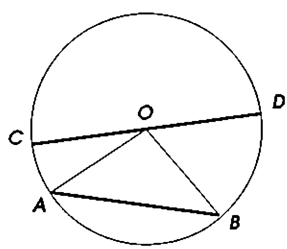

Figure 119

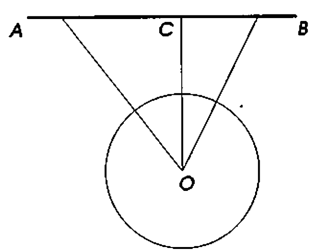

Figure 120

EXERCISES

225. A given segment is moving, remaining parallel to itself, in such a way that one of its endpoints lies on a given circle. Find the geometric locus described by the other endpoint.

226. A given segment is moving in such a way that its endpoints slide along the sides of a right angle. Find the geometric locus described by the midpoint of this segment.

227. On a chord AB, two points are taken the same distance away from the midpoint C of this chord, and through these points, two perpendiculars to AB are drawn up to their intersections with the circle. Prove that these perpendiculars are congruent.
Hint: Fold the diagram along the diameter passing through C.

228. Two intersecting congruent chords of the same circle are divided by their intersection point into respectively congruent segments.

229. In a disk, two chords CC' and DD' perpendicular to a diameter AB are drawn. Prove that the segment MM' joining the midpoints of the chords CD and $C'D'$ is perpendicular to AB.

230. Prove that the shortest of all chords, passing through a point A taken in the interior of a given circle, is the one which is perpendicular to the diameter drawn through A.

*231.** Prove that the closest and the farthest points of a given circle from a given point lie on the secant passing through this point and the center.
Hint: Apply the triangle inequality.

232. Divide a given arc into $4, 8, 16, \dots$ congruent parts.

233. Construct two arcs of the same radius, given their sum and difference.

234. Bisect a given circle by another circle centered at a given point.

235. Through a point inside a disk, draw a chord which is bisected by this point.

236. Given a chord in a disk, draw another chord which is bisected by the first one and makes a given angle with it. (Find out for which angles this is possible.)

237. Construct a circle, centered at a given point, which cuts off a chord of a given length from a given line.

238. Construct a circle of a given radius, with the center lying on one side of a given angle, and such that on the other side of the angle it cuts out a chord of a given length.

2 Relative positions of a line and a circle

112. A line and a circle can obviously be found only in one of the following mutual positions:

(1) *The distance from the center to the line is greater than the radius of the circle* (Figure 120), i.e. the perpendicular OC dropped

to the line from the center O is greater than the radius. Then the point C of the line is farther away from the center than the points of the circle and lies therefore outside the disk. Since all other points of the line are even farther away from O than the point C (slants are grater than the perpendicular), then they all lie outside the disk, and hence the line has no common points with the circle.

(2) *The distance from the center to the line is smaller than the radius* (Figure 121). In this case the point C lies inside the disk, and therefore the line and the circle intersect.

(3) *The distance from the center to the line equals the radius* (Figure 122), i.e. the point C is on the circle. Then any other point D of the line, being farther away from O than C, lies outside the disk. In this case the line and the circle have therefore only one common point, namely the one which is the foot of the perpendicular dropped from the center to the line.

Such a line, which has only one common point with the circle, is called a **tangent** to the circle, and the common point is called the **tangency point**.

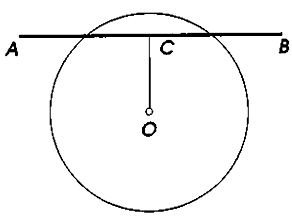

Figure 121

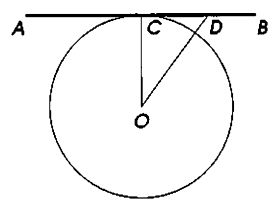

Figure 122

113. We see therefore that out of three possible cases of disposition of a line and a circle, tangency takes place only in the third case, i.e. when the perpendicular to the line dropped from the center is a radius, and in this case the tangency point is the endpoint of the radius lying on the circle. This can be also expressed in the following way:

(1) ***if a line*** (AB) ***is perpendicular to the radius*** (OC) ***at its endpoint*** (C) ***lying on the circle, then the line is tangent to the circle***, and *vice versa:*

(2) ***if a line is tangent to a circle, then the radius drawn to the tangency point is perpendicular to the line.***

114. `Problem.` *To construct a tangent to a given circle such that it is parallel to a given line AB* (Figure 123).

Drop to AB the perpendicular OC from the center, and through the point D, where the perpendicular intersects the circle, draw $EF \| AB$. The required tangent is EF. Indeed, since $OC \perp AB$ and $EF \| AB$, we have $EF \perp OD$, and a line perpendicular to a radius at its endpoint lying on the circle, is a tangent.

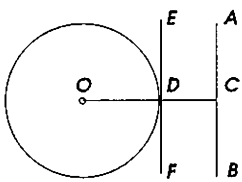

Figure 123

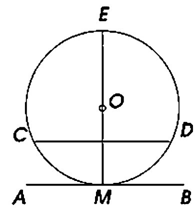

Figure 124

115. Theorem. ***If a tangent is parallel to a chord, then the tangency point bisects the arc subtended by the chord.***

Let a line AB be tangent to a circle at a point M (Figure 124) and be parallel to a chord CD; it is required to prove that $\overset{\frown}{CM} = \overset{\frown}{MD}$.

The diameter ME passing through the tangency point M is perpendicular to AB and therefore perpendicular to CD. Thus the diameter bisects the arc CMD (§105), i.e. $\overset{\frown}{CM} = \overset{\frown}{MD}$.

EXERCISES

239. Find the geometric locus of points from which the tangents drawn to a given circle are congruent to a given segment.

240. Find the geometric locus of centers of circles described by a given radius and tangent to a given line.

241. Two lines passing through a point M are tangent to a circle at the points A and B. The radius OB is extended past B by the segment $BC = OB$. Prove that $\angle AMC = 3\angle BMC$.

242. Two lines passing through a point M are tangent to a circle at the points A and B. Through a point C taken on the smaller of the arcs AB, a third tangent is drawn up to its intersection points D and E with MA and MB respectively. Prove that (1) the perimeter of $\triangle DME$, and (2) the angle DOE (where O is the center of the circle) do not depend on the position of the point C.
Hint: The perimeter is congruent to $MA+MB$; $\angle DOE = \frac{1}{2}\angle AOB$.

243. On a given line, find a point closest to a given circle.

244. Construct a circle which has a given radius and is tangent to a given line at a given point.

245. Through a given point, draw a circle tangent to a given line at another given point.

246. Through a given point, draw a circle that has a given radius and is tangent to a given line.

247. Construct a circle tangent to the sides of a given angle, and to one of them at a given point.

248. Construct a circle tangent to two given parallel lines and passing through a given point lying between the lines.

249. On a given line, find a point such that the tangents drawn from this point to a given circle are congruent to a given segment.

3 Relative positions of two circles

116. `Definitions.` Two circles are called **tangent** to each other if they have only one common point. Two circles which have two common points are said to **intersect** each other.

Two circles cannot have three common points since if they did, there would exist two circles passing through the same three points, which is impossible (§104).

We will call the **line of centers** the infinite line passing through the centers of two circles.

117. `Theorem.` ***If two circles*** (Figure 125) ***have a common point*** (A) ***situated outside the line of centers, then they have one more common point*** (A') ***symmetric to the first one with respect to the line of centers,*** (and hence such circles intersect).

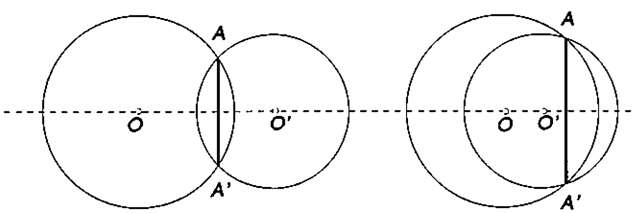

Figure 125

Indeed, the line of centers contains diameters of each of the circles and is therefore an axis of symmetry of each of them. Thus the point

A' symmetric to the common point A with respect to this axis of symmetry (and situated on the other side of it) must lie on each of these two circles.

The axis of symmetry is the perpendicular bisector of the segment AA' connecting two symmetric points A and A'. Thus we obtain:

Corollary. *The common chord* (AA', Figure 125) *of two intersecting circles is perpendicular to the line of centers and is bisected by it.*

118. Theorem. ***If two circles have a common point*** (A, Figures 126, 127) ***situated on the line of centers, then they are tangent to each other.***

The circles cannot have another common point *outside* the line of centers, because then they would also have a third common point on the other side of the line of centers, in which case they would have to coincide. The circles cannot have another common point *on* the line of centers. Indeed, then they would have two common points on the line of centers. The common chord connecting these points would have been a common diameter of the circles, and two circles with a common diameter coincide.

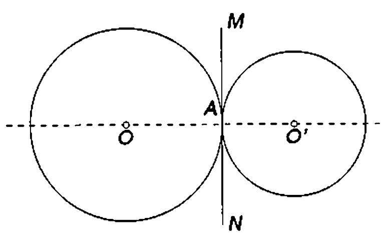

Figure 126

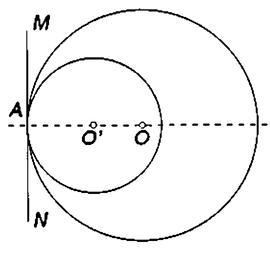

Figure 127

Remark. The tangency of two circles is called **external** if the circles are situated outside one another (Figure 126), and ***internal*** if one of them is situated inside the other (Figure 127).

119. Converse theorem. ***If two circles are tangent*** (at a point A, Figures 126, 127), ***then the tangency point lies on the line of centers.***

The point A cannot lie outside the line of centers, because otherwise the circles would have one more common point, which contradicts the hypothesis of the theorem.

Corollary. *Two tangent circles have the same tangent line at* *their tangency point*, because the line MN (Figures 126, 127) passing through the tangency point A and perpendicular to the radius OA is also perpendicular to the radius $O'A$.

120. Various cases of relative positions of two circles. Denote radii of the two circles by the letters R and R' (assuming that $R \geq R'$), and the distance between the centers by the letter d. Examine relationships between these quantities in various cases of mutual position of the circles. There are five such cases, namely:

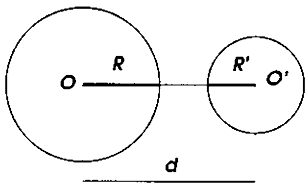

Figure 128

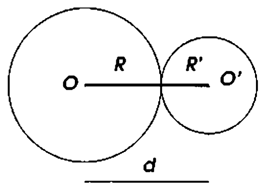

Figure 129

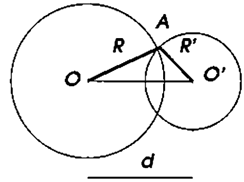

Figure 130

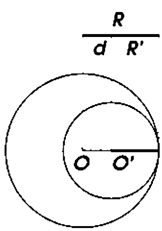

Figure 131

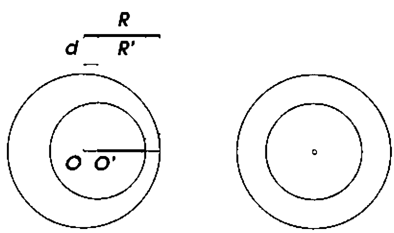

Figure 132

(1) *The circles lie outside each other without tangency* (Figure 128); in this case obviously $d > R + R'$.

(2) *The circles have an external tangency* (Figure 129); then $d = R + R'$ since the tangency point lies on the line of centers.

(3) *The circles intersect* (Figure 130); then $d < R + R'$, and at the same time $d > R - R'$, since in the triangle OAO', the side OO' congruent to d is smaller than the sum, but greater than the difference of the other two sides, congruent to the radii R and R'.

(4) *The circles have an internal tangency* (Figure 131); in this case $d = R - R'$, because the tangency point lies on the line of centers.

(5) *One circle lies inside the other without tangency* (Figure 132); then obviously $d < R - R'$. In the special case when $d = 0$, the centers

of both circles merge (such circles are called **concentric**).

Remark. We let the reader to verify the converse theorems:

(1) *If $d > R + R'$, then the circles lie outside each other.*

(2) *If $d = R + R'$, then the circles are tangent externally.*

(3) *If $d < R + R'$ and at the same time $d > R - R'$, then the circles intersect.*

(4) *If $d = R - R'$, then the circles are tangent internally.*

(5) *If $d < R - R'$, then the circles lie one inside the other.*

All these propositions are easily proved by contradiction.

121. Rotation about a point. Let a plane figure, for instance $\triangle ABC$ (Figure 133), be tied rigidly to some point O of the plane. Imagine that all points of the triangle, including its vertices, are connected by segments to the point O, and that the whole figure formed by these segments, remaining in the plane of the triangle, is moving about the point O, say, in the direction shown by the arrow. Let $A'B'C'$ be the new position occupied by the triangle ABC after some time. Since we also assume that $\triangle ABC$ does not change its shape, we have: $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$. Such a transformation of a figure in its plane is called a **rotation** about a point, and the point O itself is called the **center of rotation.** Thus, in other words, a rotation about a center O is a rigid motion of a plane figure such that the distance from each point to the center remains unchanged: $AO = A'O$, $BO = B'O$, $CO = C'O$, etc. Obviously, all points of the rotated figure describe concentric arcs with the common center at the point O, whose radii are the distances of the corresponding points from the center.

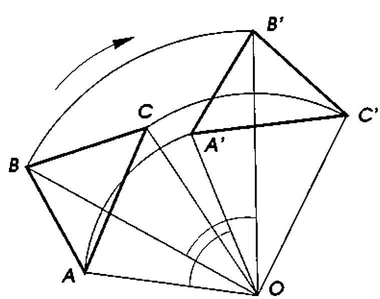

Figure 133

Notice that *central angles* (Figure 133) *corresponding to the concentric arcs, described in equal times by different points of a rotated* *figure, are congruent to each other:*

$$\angle AOA' = \angle BOB' = \angle COC' = \dots$$

Indeed, the triangles AOB and $A'OB'$ are congruent by the SSS-test, and therefore $\angle AOB = \angle A'OB'$. Adding the angle BOA' to each of them, we find: $\angle AOA' = BOB'$. Similarly one can prove that $\angle BOB' = COC'$, etc.

The common angle of rotation of all the radii is called the **rotation angle** of the figure.

Vice versa, in order to construct the rotation of a plane figure (e.g. the rotation $\triangle A'B'C'$ of $\triangle ABC$) about a given point O through a given rotation angle, it suffices to construct concentric arcs AA', BB', CC', etc., directed the same way, and corresponding to the angles $\angle AOA', \angle BOB', \angle COC', \dots$, congruent to the given rotation angle.

EXERCISES

250. Find the geometric locus of centers of circles tangent to a given circle at a given point.

251. Find the geometric locus of centers of circles described by a given radius and tangent to a given circle (consider two cases: of external and internal tangency).

252. A secant to two congruent circles, which is parallel to the line of centers OO', meets the first circle at the points A and B, and the second one at the points A' and B'. Prove that $AA' = BB' = OO'$.

253.* Prove that the shortest segment joining two non-intersecting circles lies on the line of centers.
`Hint:` Apply the triangle inequality.

254. Prove that if through an intersection point of two circles, we draw all secant segments without extending them to the exterior of the disks, then the greatest of these secants will be the one which is parallel to the line of centers.

255. Construct a circle passing through a given point and tangent to a given circle at another given point.

256. Construct a circle tangent to two given parallel lines and to a given disk lying between them.

257. Construct a circle that has a given radius, is tangent to a given disk, and passes through a given point. (Consider three cases: the given point lies (a) outside the disk, (b) on the circle, (c) inside the disk.)

4 Inscribed and some other angles

122. Inscribed angles. An angle formed by two chords drawn from the same point of a circle is called **inscribed**. Thus the angle ABC in each of Figures 134–136 is inscribed.

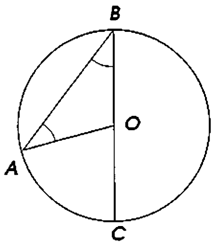

Figure 134

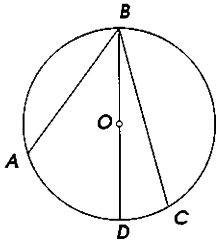

Figure 135

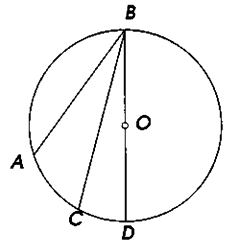

Figure 136

An angle is said to **intercept** an arc if it is contained in the interior of the angle and connects its sides. Thus the inscribed angle ABC in Figure 135 intercepts the arc ADC.

123. Theorem. ***An inscribed angle measures a half of the subtended arc.*** This theorem should be understood as follows: an inscribed angle contains as many angular degrees as a half of the arc it intercepts contains circular degrees.

In the proof of the theorem, consider the following three cases.

(1) The center O (Figure 134) lies on a side of the inscribed angle ABC. Drawing the radius AO, we obtain $\triangle AOB$ such that $OA = OB$ (as radii), and hence $\angle ABO = \angle BAO$. The angle AOC is exterior with respect to this triangle, and is congruent therefore to the sum of the angles ABO and BAO, which is twice the angle ABO. Thus the angle ABO is congruent to a half of the central angle AOC. But the angle AOC is measured by the arc AC, i.e. it contains as many angular degrees, as the arc AC contains circular degrees. Therefore the inscribed angle ABC is measured by a half of the arc AC.

(2) The center O lies in the interior of the inscribed angle ABC (Figure 135). Drawing the diameter BD we partition the angle ABC into two angles, of which (according to part (1)) one is measured by a half of the arc AD, and the other by a half of the arc DC. Thus the angle ABC is measured by the sum $\frac{1}{2}\overset{\frown}{AD} + \frac{1}{2}\overset{\frown}{DC}$, which is congruent to $\frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{DC})$, i.e. to $\frac{1}{2}\overset{\frown}{AC}$.

(3) The center O lies in the exterior of the inscribed angle ABC. Drawing the diameter BD we have

$$\angle ABC = \angle ABD - \angle CBD.$$

But the angles ABD and CBD are measured (according to part (1)) by halves of the arcs AD and CD. Therefore the angle ABC is measured by the difference $\frac{1}{2}\overset{\frown}{AD} - \frac{1}{2}\overset{\frown}{CD}$, which is congruent to $\frac{1}{2}(\overset{\frown}{AD} - \overset{\frown}{CD})$, i.e. to $\frac{1}{2}\overset{\frown}{AC}$.

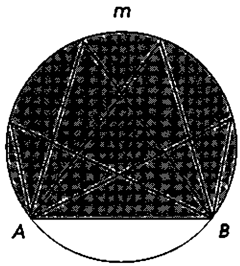

Figure 137

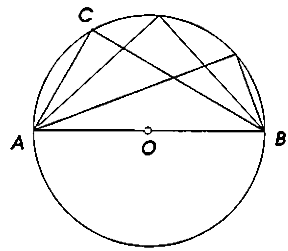

Figure 138

124. Corollaries. (1) *All inscribed angles intercepting the same arc are congruent to each other* (Figure 137), because each of them measures a half of the same arc. If the measure of one of such angles is denoted α, then one may say that the disk segment AmB **encloses the angle** α.

(2) *Any inscribed angle intercepting a diameter is right* (Figure 138), because such an angle measures a half of the semicircle, and therefore contains 90°.

125. Theorem. ***The angle*** (ACD, Figure 140) ***formed by a chord and a tangent measures a half of the intercepted arc***, (i.e. of the arc DC contained in the interior of the angle).

Let us assume first that the chord CD passes through the center O, i.e. that it is a diameter (Figure 139). Then the angle ACD is right (§113) and contains therefore 90°. But a half of the arc CmD also contains 90° since the arc CmD, being a semicircle, contains 180°. Thus the theorem holds true in this special case.

Consider now the general case when the chord CD does not pass through the center (see Figure 140, where $\angle ACD$ is acute). Drawing the diameter, CE we have:

$$\angle ACD = \angle ACE - \angle DCE.$$

The angle ACE, being the angle formed by a tangent and a diameter, measures a half of the arc CDE. The angle DCE, being inscribed, measures a half of the arc DE. Therefore the angle ACD is measured by the difference $\frac{1}{2}\,\overset{\frown}{CDE} - \frac{1}{2}\,\overset{\frown}{DE}$, i.e. by a half of the arc CD.

Similarly one can prove that an obtuse angle (BCD, Figure 140), also formed by a tangent and a chord, measures a half of the arc $CnED$. The only distinction in the proof is that this angle is not the difference, but the sum of the right angle BCE and the inscribed angle ECD.

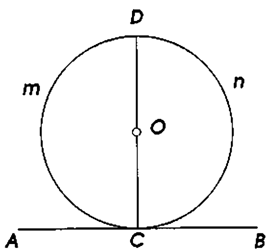

Figure 139

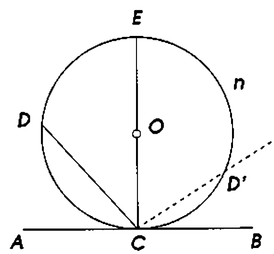

Figure 140

Remark. One may think of this theorem as a degenerate case of the previous theorem about inscribed angles. Namely, consider the angle between a tangent and a chord, e.g. $\angle BCD$ in Figure 140, and pick a point D' on the intercepted arc. Then $\angle BCD$ becomes the sum of $\angle BCD'$ and the inscribed angle $D'CD$. The arc CnD intercepted by $\angle BCD$ also becomes the sum of the corresponding arcs CD' and $D'nD$. Now let the point D' move along the circle toward the point C. When D' approaches C, the position of the secant ray CD' approaches the position of the tangent CB. Then measures of $\overset{\frown}{CD'}$ and $\angle BCD'$ both approach zero, and measures of $\overset{\frown}{D'nD}$ and $\angle D'CD$ approach those of $\overset{\frown}{CnD}$ and $\angle BCD$ respectively. Thus the property of the inscribed angle $D'CD$ to measure a half of $\overset{\frown}{D'nD}$, transforms into the property of the angle CBD between a tangent and a chord to measure a half of the intercepted arc CnD.

126. Theorem. (1) ***An angle*** (ABC, Figure 141), ***whose vertex lies inside a disk, is measured by the semisum of two arcs*** (AC and DE), ***one of which is intercepted by this angle, and the other by the angle vertical to it.***

(2) ***An angle*** (ABC, Figure 142), ***whose vertex lies outside***

a disk, and whose sides intersect the circle, is measured by the semidifference of the two intercepted arcs (AC and ED).

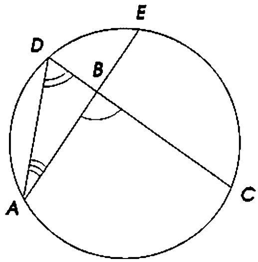

Figure 141

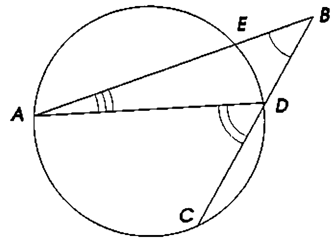

Figure 142

Drawing the chord AD (on each diagram), we obtain $\triangle ABD$ for which the angle ABC in question is exterior, when its vertex lies inside the disk, and interior, when it lies outside the disk. In the first case therefore $\angle ABC = \angle ADC + \angle DAE$, and in the second case $\angle ABC = \angle ADC - \angle DAE$. But the angles ADC and DAE, as inscribed, are measured by halves of the arcs AC and DE. Thus in the first case the angle ABC is measured by the sum $\frac{1}{2}\overset{\frown}{AC} + \frac{1}{2}\overset{\frown}{DE}$ congruent to $\frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{DE})$, and in the second case by the difference $\frac{1}{2}\overset{\frown}{AC} - \frac{1}{2}\overset{\frown}{DE}$ congruent to $\frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{DE})$.

EXERCISES

Computation problems

258. Compute the degree measure of an inscribed angle intercepting an arc congruent to $\frac{1}{12}$th part of the circle.

259. A disk is partitioned into two disk segments by a chord dividing the circle in the proportion $5:7$. Compute the angles enclosed by these segments.

260. Two chords intersect at an angle $36°15'30''$. Express in degrees, minutes, and seconds the two arcs intercepted by this angle and the angle vertical to it, if one of these arcs measures $2/3$ of the other.

261. The angle between two tangents drawn from the same point to a circle is $25°15'$. Compute the arcs contained between the tangency points.

262. Compute the angle formed by a tangent and a chord, if the chord divides the circle in the proportion $3 : 7$.

263. Two circles of the same radius intersect at the angle $2d/3$. Express in degrees the smaller of the arcs contained between the intersection points.

Remark: The angle between two intersecting arcs is defined as the angle between the tangent lines to these arcs drawn at the intersection point.

264. A tangent is drawn through one endpoint of a diameter and a secant through the other, so that they make the angle $20°30'$. Compute the smaller of the arcs contained between the tangent and the secant.

Find the geometric locus of:

265. The feet of the perpendiculars dropped from a given point A to lines passing through another given point B.

266. The midpoints of chords passing through a point given inside a disk.

267. Points from which a given circle is seen at a given angle (i.e. the angle between two tangents to the given circle drawn from the point is congruent to the given angle).

Prove theorems:

268. If two circles are tangent, then any secant passing through the tangency point cuts out on the circles opposed arcs of the same angular measure.

269. Prove that if through the tangency point of two circles two secants are drawn, then the chords connecting the endpoints of the secants are parallel.

270. Two circles intersect at the points A and B, and through A, a secant is drawn intersecting the circles at the points C and D. Prove that the measure of the angle CBD is constant, i.e. it is the same for all such secants.

271. In a disk centered at O, a chord AB is drawn and extended by the segment BC congruent to the radius. Through the point C and the center O, a secant CD is drawn, where D denotes the second intersection point with the circle. Prove that the angle AOD is congruent to the angle ACD tripled.

272. Through a point A of a circle, the tangent and a chord AB are drawn. The diameter perpendicular to the radius OB meets the tangent and the chord (or its extension) at the points C and D respectively. Prove that $AC = CD$.

273. Let PA and PB be two tangents to a circle drawn from the same point P, and let BC be a diameter. Prove that CA and OP are parallel.

274. Through one of the two intersection points of two circles, a diameter in each of the circles is drawn. Prove that the line connecting the endpoints of these diameters passes through the other intersection point.

275. A diameter AB and a chord AC form an angle of 30°. Through C, the tangent is drawn intersecting the extension of AB at the point D. Prove that $\triangle ACD$ is isosceles.

5 Construction problems

127. Problem. *To construct a right triangle given its hypotenuse a and a leg b* (Figure 143).

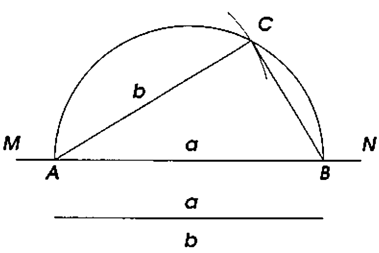

Figure 143

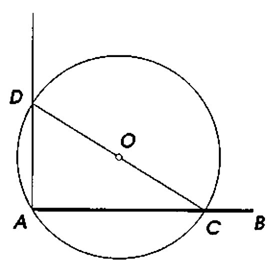

Figure 144

On a line MN, mark $AB = a$ and describe a semicircle with AB as a diameter. (For this, bisect AB, and take the midpoint for the center of the semicircle and $\frac{1}{2}AB$ for the radius.) Then draw an arc of radius congruent to b centered at the point A (or B). Connect the intersection point C of the arc and the semicircle, with the endpoints of the diameter AB. The required triangle is ABC, since the angle C is right (§124), a is the hypotenuse, and b is a leg.

128. Problem. *To erect a perpendicular to a ray AB* (Figure 144) *at the endpoint A without extending the ray beyond this point.*

Take outside the line AB any point O such that the circle, centered at O and of radius congruent to the segment OA, intersects the ray AB at some point C. Through this point C, draw the diameter CD and connect its endpoint D with A. The line AD is the required perpendicular, because the angle A is right (as inscribed intercepting a diameter).

129. Problem. *Through a given point, to draw a tangent to a given circle.*

Consider two cases:

(1) *The given point* (C, Figure 145) *lies on the circle itself.* Then draw the radius to this point, and at its endpoint C, erect the perpendicular AB to this radius (e.g. as explained in the previous problem).

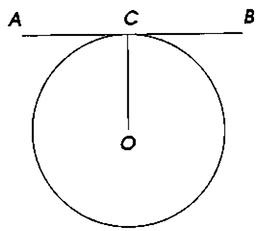

Figure 145

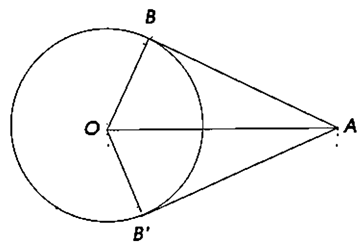

Figure 146

(2) *The given point* (A, Figure 146) *lies outside the disk bounded by the given circle.* Then, connecting A with the center O, construct the circle with AO as a diameter. Through the points B and B' at which this circle intersects the given one, draw the lines AB and AB'. These lines are the required tangents, since the angles OBA and $OB'A$ are right (as inscribed intercepting a diameter).

Corollary. *Two tangent segments, drawn to a circle from a point outside the disk bounded by it, are congruent and form congruent angles with the line connecting this point with the center.* This follows from the congruence of the right triangles OBA and $OB'A$ (Figure 146).

130. Problem. *Given two circles, to construct a common tangent* (Figure 147).

(1) **Analysis.** Suppose that the problem has been solved. Let AB be a common tangent, A and B the tangency points. Obviously, if we find one of these points, e.g. A, then we can easily find the other. Draw the radii OA and $O'B$. These radii, being perpendicular to the common tangent, are parallel to each other. Therefore, if we draw through O' the line $O'C$ parallel to BA, then $O'C$ will be perpendicular to OC. Thus, if we draw a circle of radius OC centered

at O, then $O'C$ will be tangent to it at the point C. The radius of this auxiliary circle is $OA - CA = OA - O'B$, i.e. it is congruent to the difference of the radii of the given circles.

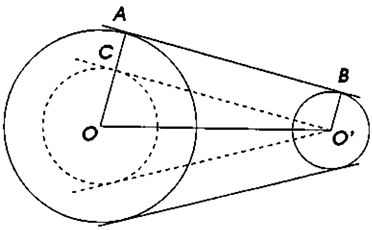

Figure 147

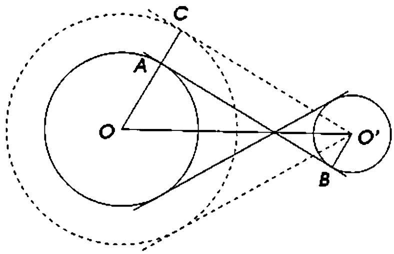

Figure 148

Construction. Thus the required construction can be performed as follows. Describe the circle centered at O of radius congruent to the difference of the given radii. From O', draw a tangent $O'C$ to this circle (as described in the previous problem). Through the point C, draw the radius OC and extend it beyond C up to the intersection point A with the given circle. Finally, through the point A, draw the line AB parallel to CO'.

Research. The construction is possible when the center O' lies in the exterior of the auxiliary circle. In this case we obtain two common tangents to the circles, each parallel to one of the two tangents from the point O' to the auxiliary circle. These two common tangents are called **external**.

For the point O' to be in the exterior of the auxiliary circle, the segment OO' has to be greater than the difference of the radii of the given circles. According to the results of §120, this is true unless one of the given disks contains the other. When one of the circles lies inside the other, obviously, no common tangent is possible. When the circles have an internal tangency, the perpendicular to the line of centers erected at the tangency point is, evidently, the only common tangent of the circles. Otherwise, i.e. when neither of the disks contains the other, there exist, as we have seen, two external common tangents.

When the two given circles do not intersect, i.e. when OO' is greater than the *sum* of the given radii, there also exist two **internal common tangents** (Figure 148) which can be constructed as follows.

(2) Analysis. Suppose that the problem has been solved, and

let AB be such a common tangent. Draw the radii OA and $O'B$ to the tangency points A and B. These radii, being perpendicular to the common tangent, are parallel to each other. Thus, if we draw from O' the line $O'C \| BA$ and extend the radius OA beyond A to its intersection with $O'C$ at the point C, then OC will be perpendicular to $O'C$. Therefore the auxiliary circle described about the center O by the radius OC will be tangent to the line $O'C$ at the point C. The radius of the auxiliary circle is $OA + AC = OA + O'B$, i.e. it is congruent to the sum of the radii of the given circles.

`Construction.` Thus the construction can be performed this way: draw the circle centered at O of radius congruent to the sum of the given radii. From the point O', draw a line $O'C$ tangent to the auxiliary circle at the point C. Connect the tangency point C with O, and through the intersection point A of OC with the circle, draw the line $AB \| CO'$.

The second internal common tangent is parallel to the other tangent from O' to the auxiliary circle and is constructed similarly.

When the segment OO' is congruent to the sum of the given radii, the two given circles have an external tangency (§120). In this case, the perpendicular to the line of centers erected at the tangency point is, evidently, the only internal common tangent of the circles. Finally, when the two disks overlap, no internal tangents exist.

131. `Problem.` *On a given segment AB, to construct a disk segment enclosing a given angle* (Figure 149).

`Analysis.` Suppose that the problem has been solved, and let AmB be a disk segment enclosing the given angle α, i.e. such that any angle ACB inscribed in it is congruent to α. Draw the auxiliary line AE tangent to the circle at the point A. Then the angle BAE formed by the tangent and the chord AB, is also congruent to the inscribed angle ACB, since both measure a half of the arc AnB. Now let us take into account that the center O of the circle lies on the perpendicular bisector DO of the chord AB, and at the same time on the perpendicular (AO) to the tangent (AE) erected at the tangency point. This suggests the following construction.

`Construction.` At the endpoint A of the segment AB, construct an angle BAE congruent to α. At the midpoint of AB erect the perpendicular DO, and at the point A, erect the perpendicular to AE. Taking the intersection point O of these perpendiculars for the center, describe the circle of radius AO.

`Proof.` Any angle inscribed into the disk segment AmB is measured by a half of the arc AnB, and the half of this arc is also the measure of $\angle BAE = \alpha$. Thus AmB is the required disk segment.

Remark. On Figure 149, the disk segment AmB enclosing the angle α, is constructed on the upper side of the line AB. Another such disk segment can be constructed symmetric to AmB about the axis AB. Thus, one could say that *the geometric locus of points, from which a given line segment AB is seen at a given angle α, consists of the arcs of two disk segments, each enclosing the given angle, which are symmetric to each other about the axis AB.*

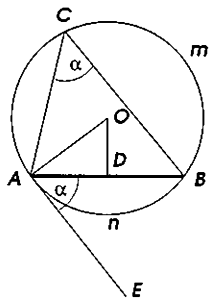

Figure 149

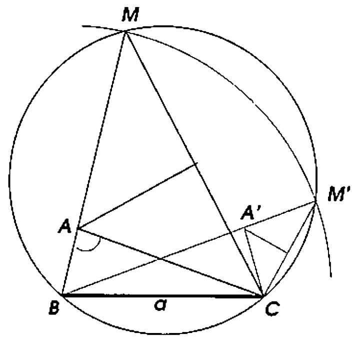

Figure 150

132. The method of geometric loci. Many construction problems can be successfully approached using the concept of geometric locus. This method, known already to *Plato* (4th century B.C.), can be described as follows. Suppose that a proposed problem consists in finding a point which has to satisfy certain conditions. Discard one of these conditions; then the problem becomes under-determined: it may admit infinitely many solutions, i.e. infinitely many points satisfying the remaining conditions. These points form a geometric locus. Construct this locus if possible. Then reinstall the previously discarded condition, but discard another one; the problem will again have infinitely many solutions which will form another geometric locus. Construct it if possible. A point satisfying all the conditions of the original problem belongs to both geometric loci, i.e. it must lie in their intersection. The construction will be possible or impossible depending on whether the loci intersect or not, and the problem will have as many solutions as there are intersection points. Let us illustrate this method by an example, which also shows that sometimes adding auxiliary lines to a diagram can be useful.

133. Problem. *To construct a triangle, given its base a, the angle at the vertex A, and the sum s of the lateral sides.*

Let ABC (Figure 150) be the required triangle. In order to add to the diagram the given sum of lateral sides, let us extend BA past A and mark on it the segment $BM = s$. Connecting M with C, we obtain an auxiliary triangle BMC. If we manage to construct this triangle, then we can easily construct the required triangle ABC. Indeed, note that the triangle CAM is isosceles ($AC = AM$), and hence A can be found as the intersection of BM with the perpendicular bisector of MC.

The construction of the triangle BMC reduces to finding the point M. Since the triangle CAM is isosceles, we have $\angle M = \angle MCA = \frac{1}{2}\angle BAC$. We see that the point M must satisfy two conditions: (1) it has distance s from B, and (2) the angle at which the segment BC is seen from M is congruent to $\frac{1}{2}\angle A$. Thus the construction of M reduces to intersecting two geometric loci such that we know how to construct each of them. The problem has no solution when these loci do not intersect, and has one or two solutions depending on whether the loci are tangent to each other or intersect. On our diagram, we obtain two (congruent!) triangles ABC and $A'BC$ satisfying the requirements of the problem.

Sometimes a problem requires finding a line (rather than a point) satisfying several conditions. Discarding one of the conditions, we will obtain infinitely many lines satisfying the remaining conditions. It may happen that all such lines can be described in terms of a certain curve (for instance, as all lines tangent to a certain circle). Discarding another condition and reinstalling the previously discarded one, we will obtain infinitely many lines again, which may define some other curve. Constructing, if possible, both curves we then determine the required line. Let us give an example.

134. Problem. *To draw a secant of two given disks O and O', so that the segments of the secant contained inside the disks are congruent respectively to two given segments a and a'.*

If we take into account only one of the requirements, for example, that the part of the secant inside the disk O is congruent to a, then we obtain infinitely many secants which have to be equidistant from the center of the disk (since congruent chords are equidistant from the center). Therefore, if we construct inside O a chord congruent to a and then describe the circle concentric to O of radius congruent to the distance from the chord to the center, then all the secants in question will be tangent to this auxiliary circle. Similarly, taking into account only the second condition, we will see that the required secant must be tangent to the second auxiliary circle concentric to

O'. Thus the problem reduces to constructing a common tangent to two circles.

EXERCISES

Prove theorems

276. Given two circles with external tangency, prove that the common tangent passing through the tangency point, bisects the segments of external common tangents bounded by the tangency points.

277. To two circles tangent externally at a point A, a common external tangent BC is drawn (where B and C are the tangency points). Prove that the angle BAC is right.
Hint: Draw through A a common tangent and examine the triangles ABD and ADC.

Construction problems

278. Given two points, construct a line such that the perpendiculars dropped from these points to this line have given lengths.

279. Construct a line making a given angle with a given line and tangent to a given circle. (How many solutions are there?)

280. From a point outside a disk, construct a secant such that its segment inside the disk is congruent to a given segment.

281. Construct a circle that has a given radius, and is tangent to a given line and a given circle.

*282.** Construct a circle tangent to a given line and tangent to a given circle at a given point (two solutions).

283. Construct a circle tangent to a given circle and tangent to a given line at a given point (two solutions).

284. Construct a circle that has a given radius and cuts out chords of given lengths on the sides of a given angle.

285. Construct a disk tangent to two given disks, and to one of them at a given point. (Consider three cases: the required disk contains (1) both given disks, (2) one of them, (3) none of them.)

286. Construct a circle tangent (externally or internally) to three given congruent circles.

*287.** Into a given circle, inscribe three congruent disks tangent to each other and to the given circle.

*288.** Through a given point inside a disk, draw a chord such that the difference of its segments is congruent to a given segment.

Hint: Draw the concentric circle passing through the given point, and construct in this circle a chord of the given length.

289. Through an intersection point of two circles, draw a secant such that its segment inside the given disks is congruent to a given length.
Hint: Construct a right triangle whose hypotenuse is the segment between the centers of the given disks, and one of the legs is congruent to a half of the given length.

290. From a point outside a disk, draw a secant ray such that its external and internal parts are congruent.
Hint: Let O be the center of the disk, R its radius, and A the given point. Construct $\triangle AOB$, where $AB = R$, $OB = 2R$. If C is the midpoint of the segment OB, then the line AC is the required one.

291. Construct a circle tangent to two given non-parallel lines (1) if the radius is given, (2) if instead one of the tangency points is given.

292. On a given line, find a point from which a given segment is seen at a given angle.

293. Construct a triangle, given its base, the angle at the vertex, and the altitude.

294. Construct a triangle, given one of its angles and two of its altitudes, one of which is drawn from the vertex of the given angle.

295. Construct a tangent to the arc of a given sector such that the segment of the tangent between the extensions of the radii bounding the sector is congruent to a given segment.
Hint: Reduce the problem to the previous one.

296. Construct a triangle, given its base, the angle at the vertex; and the median bisecting the base.

297. Given the positions of two segments a and b in the plane, find a point from which the segment a is seen at a given angle α, and the segment b at a given angle β.

298. In a given triangle, find a point from which its sides are seen at the same angle.

*299.** Construct a triangle, given its angle at the vertex, and the altitude and the median drawn to the base.
Hint: Double the median extending it past the base, connect the endpoint with the vertices at the base, and consider the parallelogram thus formed.

*300.** Construct a triangle, given its base, an angle adjacent to the base, and the angle between the median drawn from the vertex of the first given angle and the side to which this median is drawn.

301. Construct a parallelogram, given its diagonals and an angle.

*302.** Construct a triangle, given its base, its angle at the vertex, and the sum or the difference of the other two sides.

303. Construct a quadrilateral, given its diagonals, two adjacent sides, and the angle between the two remaining sides.

*304.** Given three points A, B, and C, construct a line passing through A such that the distance between the perpendiculars to this line dropped from the points B and C is congruent to a given segment.

6 Inscribed and circumscribed polygons

135. Definitions. If all vertices of a polygon ($ABCDE$, Figure 151) lie on a circle, then the polygon is called **inscribed** into the circle, and the circle is called **circumscribed** about the polygon.

If all sides of a polygon ($MNPQ$, Figure 151) are tangent to a circle, then the polygon is called **circumscribed** about the circle, and the circle is called **inscribed** into the polygon.

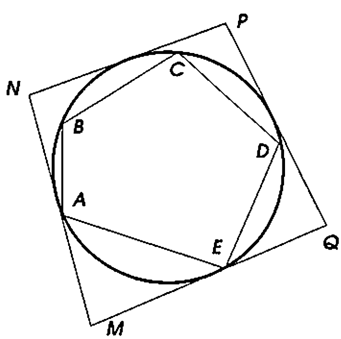

Figure 151

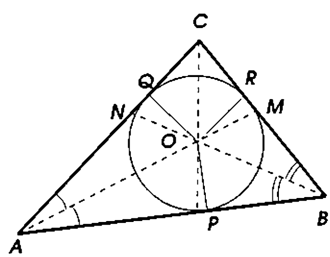

Figure 152

136. Theorems. (1) ***About any triangle, a circle can be circumscribed, and such a circle is unique.***

(2) ***Into any triangle, a circle can be inscribed, and such a circle is unique.***

(1) Vertices A, B, and C of any triangle are non-collinear. As we have seen in §104, any three such points lie on a circle, and such a circle is unique.

(2) If a circle tangent to all sides of a triangle ABC exists (Figure 152), then the center must be a point equidistant from these sides.

Let us prove that such a point exists. The geometric locus of points equidistant from the sides AB and AC is the bisector AM of the angle A (§58). The geometric locus of points equidistant from the sides BA and BC is the bisector BN of the angle B. These two bisectors will, evidently, intersect inside the triangle at some point O. This point will be equidistant from all the sides of the triangle, since it lies in both geometric loci. Thus, in order to inscribe a circle into a triangle, bisect two of its angles, say A and B, take the intersection point of the bisectors for the center, and take for the radius any of the perpendiculars OP, OQ, or OR, dropped from the center to the sides of the triangle. The circle will be tangent to the sides at the points P, Q, and R, since at these points the sides are perpendicular to the radii at their endpoints lying on the circle (§113). Another such an inscribed circle cannot exist, since two bisectors can intersect only at one point, and from a point only one perpendicular to a line can be dropped.

Remark. We leave it to the reader to verify that the center of the circumscribed circle lies inside the triangle if and only if the triangle is scalene. For an obtuse triangle, the center lies outside it, and for a right triangle at the midpoint of the hypotenuse. The center of the inscribed circle always lies inside the triangle.

Corollary. The point O (Figure 152), being equidistant from the sides CA and CB, must lie on the bisector of the angle C. Therefore *bisectors of the three angles of a triangle intersect at one point.*

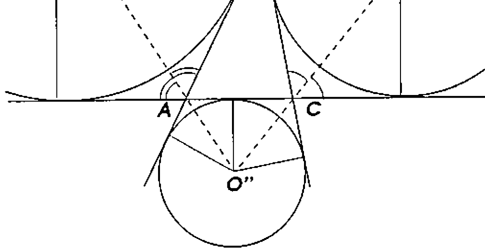

Figure 153

137. Exscribed circles. The circles tangent to one side of a triangle and to the extensions of two other sides (such circles lie outside the triangle, Figure 153) are called **exscribed**. Each triangle has three such circles. To construct them, draw bisectors of the exterior angles of the triangle ABC, and take their intersection points for the centers. Thus, the center of the circle inscribed into the angle A, is the point O, i.e. the intersection point of the bisectors BO and CO of the exterior angles not supplementary to A. The radius of this circle is the perpendicular dropped from O to any of the sides of the triangle.

138. Inscribed quadrilaterals. (1) ***In a convex inscribed quadrilateral, the sum of opposite angles is congruent to two right angles.***

(2) Conversely, ***if a convex quadrilateral has the sum of opposite angles congruent to two right angles, then it can be circumscribed by a circle.***

(1) Let $ABCD$ (Figure 154) be an inscribed convex quadrilateral; it is required to prove that

$$\angle B + \angle D = 2d \text{ and } \angle A + \angle C = 2d.$$

Since the sum of all the four angles of any convex quadrilateral is $4d$ (§82), then it suffices to prove only one of the required equalities. Let us prove for example that $\angle B + \angle D = 2d$.

The angles B and D, as inscribed, are measured: the former by a half of the arc ADC, and the latter by a half of the arc ABC. Therefore the sum $\angle B + \angle D$ is measured by the sum $\frac{1}{2}\overset{\frown}{ADC} + \frac{1}{2}\overset{\frown}{ABC}$, which is congruent to $\frac{1}{2}(\overset{\frown}{ADC} + \overset{\frown}{ABC})$, i.e. a half of the whole circle. Thus $\angle B + \angle D = 180° = 2d$.

(2) Let $ABCD$ (Figure 154) be a convex quadrilateral such that $\angle B + \angle D = 2d$, and therefore $\angle A + \angle C = 2d$. It is required to prove that a circle can be circumscribed about such a quadrilateral.

Through any three vertices of it, say through A, B, and C, draw a circle (which is always possible). The fourth vertex D must lie on this circle. Indeed, if it didn't, it would lie either inside the disk, or outside it. In either case the angle D would not measure a half of the arc ABC, and therefore the sum $\angle B + \angle D$ would not measure the semisum of the arcs ADC and ABC. Thus this sum would differ from $2d$, which contradicts the hypothesis.

`Corollaries.` (1) *Among all parallelograms, rectangles are the only ones which can be circumscribed by a circle.*

(2) *A trapezoid can be circumscribed by a circle only if it is isosceles.*

139. Circumscribed quadrilaterals. ***In a circumscribed quadrilateral, the sums of opposite sides are congruent.***

Let $ABCD$ (Figure 155) be a circumscribed quadrilateral, i.e. the sides of it are tangent to a circle. It is required to prove that $AB + CD = BC + AD$.

Denote the tangency points by the letters M, N, P, and Q. Since two tangents drawn from the same point to a circle are congruent, we have $AM = AQ$, $BM = BN$, $CN = CP$, and $DP = DQ$. Therefore

$$AM + MB + CP + PD = AQ + QD + BN + NC,$$
$$\text{i.e.}\quad AB + CD = AD + BC.$$

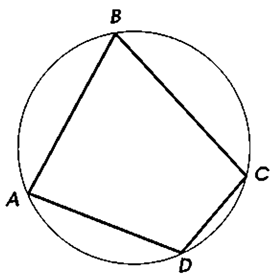

Figure 154

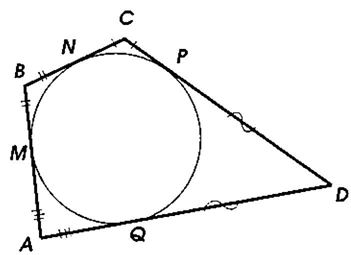

Figure 155

EXERCISES

305. Into a given circle, inscribe a triangle whose angles are given.

306. About a given circle, circumscribe a triangle whose angles are given.

307. Construct a triangle, given the radius of its inscribed circle, the angle at the vertex, and the altitude.

308. Into a given circle, inscribe a triangle, given the sum of two of its sides and the angle opposite to one of them.

309. Into a given circle, inscribe a quadrilateral, given one of its sides, and both angles not adjacent to it.

310. Inscribe a circle into a given rhombus.

311. Into a given sector, inscribe a circle tangent to the radii and the arc bounding the sector.

*312.** Into an equilateral triangle, inscribe three disks which are pairwise tangent to each other, and each of them is tangent to two sides of the triangle.

313. Construct a quadrilateral assuming that it can be circumscribed by a circle, and that three of its sides and a diagonal are given.

314. Construct a rhombus, given its side and the radius of the inscribed circle.

315. Circumscribe an isosceles right triangle about a given circle.

316. Construct an isosceles triangle, given its base and the radius of the inscribed circle.

*317.** Through two given points on a circle, construct two parallel chords with a given sum.

*318.** On a circle circumscribed about an equilateral $\triangle ABC$, a point M is taken. Prove that the greatest of the segments MA, MB, MC is congruent to the sum of the other two.

*319.** The feet of perpendiculars dropped from a point of a circle to the sides of an inscribed triangle lie on the same line (called **Simson's line**).

Hint: A proof is based on properties of inscribed angles (§123), and angles of inscribed quadrilaterals (§138).

7 Four concurrency points in a triangle

140. We have seen that:

(1) *the three perpendicular bisectors to the sides of a triangle intersect at one point* (which is the center of the circumscribed circle and is often called the **circumcenter** of the triangle);

(2) *the three bisectors of the angles of a triangle intersect at one point* (which is the center of the inscribed circle, and often called **incenter** of the triangle).

The following two theorems point out two more remarkable points in a triangle: (3) the intersection point of the three altitudes, and (4) the intersection point of the three medians.

141. Theorem. ***Three altitudes of a triangle intersect at one point.***

Through each vertex of $\triangle ABC$ (Figure 156), draw the line parallel to the opposite side of the triangle. Then we obtain an auxiliary triangle $A'B'C'$ whose sides are perpendicular to the altitudes of the given triangle. Since $C'B = AC = BA'$ (as opposite sides of parallelograms), then the point B is the midpoint of the side $A'C'$. Similarly,

C is the midpoint of $A'B'$ and A of $B'C'$. Thus the altitudes AD, BE, and CF of $\triangle ABC$ are perpendicular bisectors to the sides of $\triangle A'B'C'$, and such perpendiculars, as we know from §104, intersect at one point.

Remark. The point where the three altitudes of a triangle intersect is called its **orthocenter**. The reader may prove that the orthocenter of an acute triangle lies inside the triangle, of an obtuse triangle outside it, and for a right triangle coincides with the vertex of the right angle.

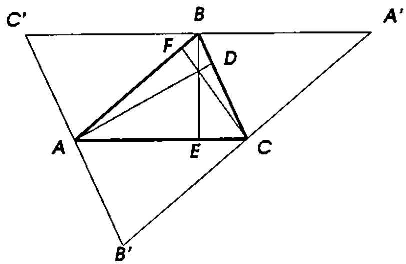

Figure 156

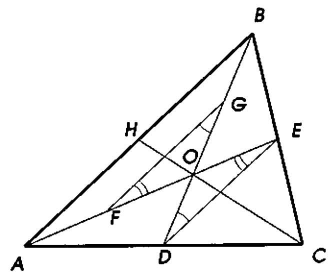

Figure 157

142. Theorem. ***The three medians of a triangle intersect at one point; this point cuts a third part of each median measured from the corresponding side.***

In $\triangle ABC$ (Figure 157), take any two medians, e.g. AE and BD, intersecting at a point O, and prove that

$$OD = \frac{1}{3}BD, \quad \text{and} \quad OE = \frac{1}{3}AE.$$

For this, bisect OA and OB at the points F and G and consider the quadrilateral $DEGF$. Since the segment FG connects the midpoints of two sides of $\triangle ABO$, then $FG \| AB$ and $FG = \frac{1}{2}AB$. The segment DE, too, connects the midpoints of two sides of $\triangle ABC$, and hence $DE \| AB$ and $DE = \frac{1}{2}AB$. From this we conclude that $DE \| FG$ and $DE = FG$, and therefore the quadrilateral $DEGF$ is a parallelogram (§86). It follows that $OF = OE$ and $OD = OG$, i.e. that $OE = \frac{1}{3}AE$ and $OD = \frac{1}{3}BD$.

If we consider now the third median and one of the medians AE or BD, then we similarly find that their intersection point cuts from each of them a third part measured from the foot. Therefore the third median must intersect the medians AE and BD at the very same point O.

Remarks. (1) It is known from physics that the intersection point of the medians of a triangle is the **center of mass** (or **centroid**) of it, also called **barycenter**; it always lies inside the triangle.

(2) Three (or more) lines intersecting at one point are called **concurrent**. Thus we can say that the orthocenter, barycenter, incenter and circumcenter of a triangle are concurrency points of its altitudes, medians, angle bisectors, and perpendicular bisectors of its sides respectively.

EXERCISES

320. Construct a triangle, given its base and two medians drawn from the endpoints of the base.

321. Construct a triangle, given its three medians.

322. Into a given circle, inscribe a triangle such that the extensions of its angle bisectors intersect the circle at three given points.

323. Into a given circle, inscribe a triangle such that the extensions of its altitudes intersect the circle at three given points.

*324.** Construct a triangle given its circumscribed circle and the three points on it at which the altitude, the angle bisector and the median, drawn from the same vertex, intersect the circle.

*325.** Prove that connecting the feet of the altitudes of a given triangle, we obtain another triangle for which the altitudes of the given triangle are angle bisectors.

*326.** Prove that the barycenter of a triangle lies on the line segment connecting the circumcenter and the orthocenter, and that it cuts a third part of this segment measured from the circumcenter.
Remark: This segment is called **Euler's line** of the triangle.

*327.** Prove that for every triangle, the following nine points lie on the same circle (called **Euler's circle**, or the **nine-point circle** of the triangle): three midpoints of the sides, three feet of the altitudes, and three midpoints of the segments connecting the orthocenter with the vertices of the triangle.

*328.** Prove that for every triangle, the center of Euler's circle lies on Euler's line and bisects it.
Remark: Moreover, according to **Feuerbach's theorem**, *for every triangle, the nine-point circle is tangent to the inscribed and all three exscribed circles.*

Chapter 3

SIMILARITY

1 Mensuration

143. The problem of mensuration. So far, comparing two segments, we were able to determine if they are congruent, and if they are not then which of them is greater (§6). We have encountered this task when studying relationships between sides and angles of triangles (§§44, 45), the triangle inequality (§§48–50), and some other topics (§§51–53, 109–111, 120). Yet such comparison of segments does not provide an accurate idea about their magnitudes.

Now we pose the problem of establishing precisely the concept of length of segments and expressing lengths by means of numbers.

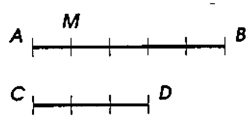

Figure 158

144. A common measure of two segments is a third segment such that it is contained in each of the first two a whole number of times with no remainder. Thus, if a segment AM (Figure 158) is contained 5 times in AB and 3 times in CD, then AM is a common measure of AB and CD. One can similarly talk about common measures of two arcs of the same radius, of two angles, and more generally of any two quantities of the same denomination.

Evidently, if the segment AM is a common measure of the segments AB and CD, then dividing AM into 2, 3, 4, etc. congruent parts we obtain smaller common measures of the same segments. Therefore, if two segments have a common measure, one can say that they have infinitely many common measures. One of them will be the greatest.

145. The greatest common measure. Finding the greatest common measure of two segments is done by the method of **consecutive exhaustion**, quite similar to the method of consecutive division which is used in arithmetic for finding the greatest common factor of two whole numbers. The method (also called the **Euclidean algorithm**) is based on the following general facts.

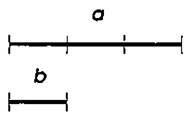

Figure 159

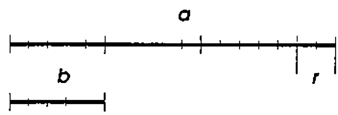

Figure 160

(1) ***If the smaller one of two segments*** (a and b, Figure 159) ***is contained in the greater one a whole number of times with no remainder, then the greatest common measure of the two segments is the smaller segment.***

Let a segment b be contained in a segment a exactly, say, 3 times. Since b is, of course, contained in itself once, then b is a common measure of a and b. This common measure is the greatest since no segment greater than b can be contained in b a whole number of times.

(2) ***If the smaller one of two segments*** (b in Figure 160) ***is contained in the greater one*** (a) ***a whole number of times with some remainder*** (r), ***then the greatest common measure of these segments*** (if it exists) ***must be the greatest common measure of the smaller segment*** (b) ***and the remainder*** (r).

Let, for instance,

$$a = b + b + b + r.$$

We can derive from this equality two conclusions:

(i) If there exists a segment fitting some number of times (i.e. without remainder) into b and some number of times into r, then it also fits a whole number of times into a. For instance, if some segment is contained in b exactly 5 times, and in r exactly 2 times, then it is contained in a exactly $5 + 5 + 5 + 2 = 17$ times.

(ii) Conversely, if there exists a segment fitting several times, without remainder, into a and b, then it also fits without remainder into r. For example, if some segment is contained in a exactly 17 times, and in b exactly 5 times, then it is contained exactly 15 times in that part of the segment a which is congruent to $3b$. Therefore in the remaining part of a, i.e. in r, it is contained $17 - 15 = 2$ times exactly.

Thus the two pairs of segments: a and b, and b and r, have the same common measures (if they exist), and therefore their greatest common measures also have to be the same.

These two theorems should also be supplemented by the following **Archimedes' axiom:**

However long is the greater segment (a), ***and however short is the smaller one*** (b), ***subtracting consecutively*** 1,2,3, ***etc. times the smaller segment from the greater one, we will always find that after some m-th subtraction, either there is no remainder left, or there is a remainder which is smaller than the smaller segment*** (b). In other words, it is always possible to find a sufficiently large whole number m such that either $mb = a$, or $mb < a < (m+1)b$.

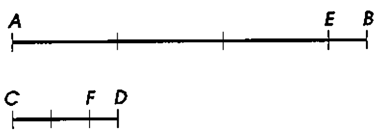

Figure 161

146. The Euclidean algorithm. Suppose it is required to find the greatest common measure of two given segments AB and CD (Figure 161).

Using a compass, exhaust the greater segment by marking on it the smaller one as many times as possible. According to Archimedes' axiom, one of two outcomes will occur: either (1) CD will fit into AB several times with no remainder, and then according to the 1st theorem the required measure will be CD, or (2) there will be a remainder EB smaller than CD (as in Figure 161). According to the second theorem, the problem will then reduce to finding the greatest common measure of the two smaller segments, namely CD and the remainder EB. To find it, do as before, i.e. exhaust CD by marking on it EB as many times as possible. Again, one of two outcomes will occur: either (1) EB will fit into CD several times with no remainder, and then the required measure will be EB, or (2) there will be a remainder FD smaller than EB (as in Figure 161). The problem is then reduced to finding the greatest common measure of another pair of smaller segments, namely EB and the second remainder FD.

Continuing this process further, we can encounter one of the following two cases:

(i) after some exhaustion step there will be no remainder left, or

(ii) the process of consecutive exhaustion will continue indefinitely (assuming that we can mark segments as small as desired, which is possible, of course, only theoretically).

In the former case, the last remainder will be the greatest common measure of the given segments. One can similarly find the greatest common measure of two arcs of the same radius, of two angles, etc.

In the latter case, the given segments cannot have any common measure. To see this, let us assume that the given segments AB and CD have a common measure. This measure, as we have seen, must be contained a whole number of times not only in AB and CD, but also in the remainder EB, and therefore in the second remainder FD, and in the third, and in the fourth, and so on. Since these remainders become smaller and smaller, each of them will contain the common measure fewer times than the previous one. For instance, if EB contains the common measure 100 times (in general m times), then FD contains it fewer than 100 times, i.e. 99 at most. The next remainder contains it fewer than 99 times, i.e. 98 at most, and so on. Since the decreasing sequence of positive whole numbers: $100, 99, 98, \dots$ (in general $m, m-1, m-2, \dots$) terminates (however large m is), then the process of consecutive exhaustion must terminate as well, i.e. no remainder will be left. Thus, if the process of consecutive exhaustion never ends, then the given segments cannot have a common measure.

147. Commensurable and incommensurable segments. Two segments are called commensurable if they have a common measure, and incommensurable if such a common measure does not exist.

Existence of incommensurable segments cannot be discovered experimentally. In the process of endless consecutive exhaustion we will always encounter a remainder so small that it will *appear* to fit the previous remainder a whole number of times: limitations of our instruments (compass) and our senses (vision) will not allow us to determine if there is any remainder left. However, incommensurable segments do exist, as we will now *prove*.

148. Theorem. ***The diagonal of a square is incommensurable to its side.***

Since the diagonal divides the square into two isosceles right triangles, then this theorem can be rephrased this way: ***the hypotenuse of an isosceles right triangle is incommensurable to its leg.***

Let us prove first the following property of such a triangle: *if we mark on the hypotenuse AC (Figure 162) of $\triangle ABC$ the segment AD congruent to the leg, and draw $DE \perp AC$, then the right triangle DEC thus formed will be isosceles, and the part BE of the leg BC will be congruent to the part DC of the hypotenuse.*

To prove this, draw the line BD and consider angles of the triangles DEC and BED. Since the triangle ABC is right and isosceles, then $\angle 1 = \angle 4$, and therefore $\angle 1 = 45°$. Therefore in the right triangle DEC we have $\angle 2 = 45°$ too, so that $\triangle DEC$ has two congruent angles, and hence two congruent sides DE and DC.

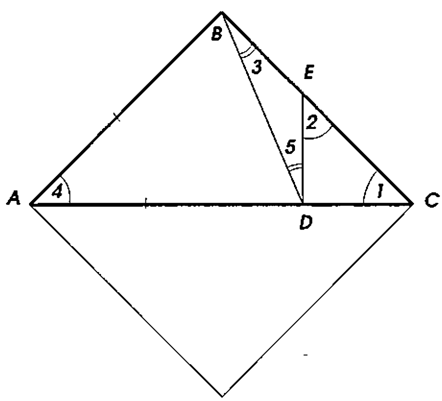

Figure 162

Furthermore, in the triangle BED, the angle 3 is congruent to the right angle B minus the angle ABD, and the angle 5 is congruent to the right angle ADE less the angle ADB. But $\angle ADB = \angle ABD$ (since $AB = AD$), and hence $\angle 3 = \angle 5$. Then the triangle BED must be isosceles, and therefore $BE = DE = DC$.

Having noted this, let us apply the Euclidean algorithm to the segments AB and AC.

Since $AC > AB$ and $AC < AB + BC$, i.e. $AC < 2AB$, then the leg AB fits the hypotenuse AC only once, and the remainder is DC. Now we have to use the remainder DC to exhaust AB, or equivalently, BC. But the segment BE is congruent to DC by the above observation. Therefore we need to further mark DC of EC. But EC is the hypotenuse of the isosceles right triangle DEC. Therefore the Euclidean algorithm now reduces to exhausting the hypotenuse EC of an isosceles right triangle by its leg DC. In its turn, this process will reduce to exhausting the hypotenuse of a new, smaller isosceles right triangle by its leg, and so on, indefinitely. Obviously, this process never ends, and therefore a common measure of the segments AC and AB does not exist.

149. Lengths of segments. The length of a segment is expressed by a number obtained by comparing this segment with another one, called the **unit of length**, such as e.g. *meter*, *centimeter*, *yard*, or *inch*.

Suppose we need to measure a given segment a (Figure 163) using a unit b, commensurable with a. If the greatest common measure of a and b is the unit b itself, then the length of a is expressed by a whole number. For instance, when b is contained in a three times, one says that the length of a is equal to 3 units (i.e. $a = 3b$). If the greatest common measure of a and b is a part of b, then the length is expressed by a fraction. For example, if $\frac{1}{4}b$ is a common measure, and it is contained in a nine times, then one says that the length of a is equal to 9/4 units (i.e. $a = \frac{9}{4}b$).

Whole numbers and fractions are called **rational numbers**.

Thus, *the length of a segment commensurable with a unit of length is expressed by a rational number telling us how many times some fraction of the unit is contained in the given segment.*

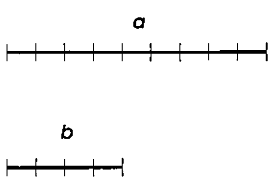

Figure 163

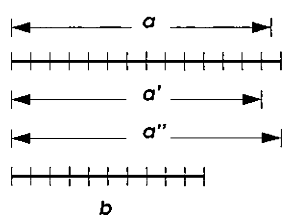

figure 164

150. Approximations. The discovery of incommensurable segments was made by ancient Greeks. It shows that rational numbers are, generally speaking, insufficient for expressing lengths of segments. For instance, according to §148, no rational number can express the length of the diagonal of a square, when its side is taken for the unit of length.

Measuring a segment a incommensurable with the unit b is done indirectly: instead of the segment a, one measures other segments commensurable with the unit and such that they differ from a by as little as one wishes. Namely, suppose we want to find commensurable segments that would differ from a by less than $\frac{1}{10}b$. Then divide the unit into 10 equal parts (Figure 164) and repeat one such part as many times as needed to exhaust a. Suppose $\frac{1}{10}b$ is contained in a thirteen times with a remainder smaller than $\frac{1}{10}b$. We obtain a segment a' commensurable with b and smaller than a. Adding $\frac{1}{10}b$ once more, we obtain another segment a'' also commensurable with b and greater than a. The lengths of the segments a' and a'' are expressed by the fractions 13/10 and 14/10. These numbers are considered as **approximations** to the length of the segment a, the first **from below**, the second **from above**. Since they both differ from a by less than $\frac{1}{10}$th of the unit, one says that each of them expresses the length with the **precision** of up to $\frac{1}{10}$ (or with the **error** smaller than $\frac{1}{10}$).

In general, to approximate the length of a segment a with the precision of up to $\frac{1}{n}$th of a unit b, one divides the unit into n equal parts and finds how many times the $\frac{1}{n}$th part of the unit is contained in a. If it is contained m times with a remainder smaller than $\frac{1}{n}b$, then the rational numbers $\frac{m}{n}$ and $\frac{m+1}{n}$ are said to approximate the length of a with the precision of up to $\frac{1}{n}$, the first from below, and the second from above.

151. Irrational numbers. The precise length of a segment incommensurable with the unit of length is expressed by an **irrational number.** [1] It can be represented by an **infinite decimal fraction** constructed as follows. One consecutively computes approximations from below for the length of the segment a with the precision of up to 0.1, then up to 0.01, then up to 0.001, and continues this process indefinitely, each time improving the precision 10 times. This way, one obtains decimal fractions first with one place after the decimal

[1] The first definition of irrational numbers, usually attributed to a Greek mathematician Eudoxus (408 – 355 B.C.), is found in Book 5 of Euclid's "Elements." Given a segment incommensurable with the unit of length, all segments commensurable with the unit (and respectively all fractions m/n expressing their lengths) are partitioned into two disjoint groups: those which are smaller than the given segment, and those which are greater. According to Eudoxus, an irrational number *is* such a partition (a *cut*, in the modern terminology) of the set of all rational numbers. This somewhat abstract construction coincides with one of the modern definitions of irrational numbers proposed by R. Dedekind [2] in the late 19th century.

point, then with two, then with three, and further on with more and more decimal places.

The result of this infinite process is an infinite decimal fraction. It cannot be written, of course, on a page since the number of decimal places is infinite. Nevertheless, an infinite decimal fraction is considered known when a rule which determines any finite number of its decimal signs is known.

Thus, *the length of a segment incommensurable with the unit of length is expressed by an infinite decimal fraction whose finite parts express lengths of segments commensurable with the unit and approximating the given segment with the errors that become consecutively smaller than* 1/10*th part of the unit,* 1/100*th,* 1/1000*th, and so on.*

152. Remarks. (1) The same infinite decimal fraction can be obtained by using approximations to the irrational number from above rather than from below. Indeed, two approximations taken with the same precision, one from above, the other from below, differ only in the rightmost decimal place. When the precision improves, the rightmost place moves farther and farther to the right, thus leaving behind the same sequence of decimal signs in both fractions.

(2) The same method of decimal approximations applies to a segment commensurable with the unit of length. The result will be the rational number, expressing the length of the segment and represented as an (infinite) decimal fraction. It is not hard to show that the decimal fraction representing a rational number is **repeating**, i.e. it contains a finite sequence of decimal signs which begins to repeat again and again starting from some decimal place and going all the way to the right. Conversely, every repeating decimal fraction, as it is not hard to see, represents a rational number. Therefore the decimal fraction representing an irrational number (e.g. the length of any segment incommensurable with the unit) is **non-repeating**. For example, the decimal fraction

$$\sqrt{2} = 1.4142\ldots$$

is non-repeating, since the number $\sqrt{2}$, as it is well known, is irrational.

(3) Rational and irrational numbers are called **real numbers**. Thus, infinite decimal fractions, repeating and non-repeating, represent (positive) real numbers.

153. The number line. The correspondence between segments and real numbers expressing their lengths allows one to represent real numbers as points on a straight line. Consider a ray OA (Figure

165) and mark on it a point B such that the segment OB is congruent to the unit of length. Every point C on the ray determines the segment OC whose length with respect to the unit OB is expressed by a positive real number c. One says that the point C represents the number c on the number line. Conversely, given a positive real number, say $\sqrt{2}$, its finite decimal approximations 1.4, 1.41, 1.414, etc. are lengths of certain segments OD_1, OD_2, OD_3, etc. commensurable with the unit. The infinite sequence of such segments approximates from below a certain segment OD. One says that the number ($\sqrt{2}$ in this example) is represented by the point D on the number line.

In particular, the point B represents the number 1, and the point O the number 0.

Now we extend the ray OA to the whole straight line. Then the point C' on the ray OA' (Figure 165), symmetric with respect to the center O to a point C on the ray OA, is said to represent the **negative real number** $-c$, i.e. the opposite to that positive number which is represented by the symmetric point C.

Thus, all real numbers: positive, zero, or negative, are represented by points on the number line. conversely, picking on any straight line any two points O and B to represent the numbers 0 and 1 respectively, we establish a correspondence between all points of the line and all real numbers.

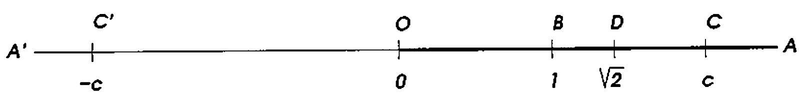

Figure 165

154. Ratio of two segments. The **ratio** of one line segment to another is defined as the positive real number which expresses the length of the first segment when the second one is taken for the unit of length. For example, if two segments a and c are such that $a = 2.1c$, i.e. if the segment a, measured by the unit c, has the length 2.1, then 2.1 is the ratio of a to c.

If both segments a and c are measured by the same unit b, then the ratio of a to c can be obtained by dividing the number expressing the length of a by the number expressing the length of c. For instance, if the lengths of a and c turned out to be 7/2 and 5/3, we can write: $a = \frac{7}{2}b$ and $c = \frac{5}{3}b$. Taking then c for the unit, we find that $b = \frac{3}{5}c$, and respectively

$$a = \frac{7}{2}b = \frac{7}{2}\left(\frac{3}{5}c\right) = \left(\frac{7}{2} \times \frac{3}{5}\right)c = \left(\frac{7}{2} : \frac{5}{3}\right)c.$$

Therefore the ratio of a to c, i.e. the length of the segment a measured by the unit c, is equal to the quotient $\frac{7}{2} : \frac{5}{3} = \frac{7\times 3}{2\times 5} = \frac{21}{10} = 2.1$.

The ratio of two segments is usually denoted as $a : c$ or $\frac{a}{c}$. Due to the property of the ratio described above, the letters a and c in these formulas can also be understood as *numbers* measuring the corresponding segments by the same unit b.

155. Proportions. A proportion expresses equality of two ratios. For instance, if it is known that the ratio $a : b$ of two segments is equal to the ratio $a' : b'$ of two other segments, then this fact can be expressed as a proportion: $a : b = a' : b'$, or

$$\frac{a}{b} = \frac{a'}{b'}.$$

In this case we will also say that the two pairs of segments: a and b, and a' and b', are **proportional** to each other.

When such pairs of segments are proportional, i.e. $a : b = a' : b'$, then $a : a' = b : b'$, i.e. the pairs a and a', and b and b' (obtained from the original ones by transposing the **mean terms** b and a') are proportional too.

Indeed, replacing the four segments with numbers that express their lengths measured with the same unit, we see that each of the resulting numerical proportions:

$$\frac{a}{b} = \frac{a'}{b'} \text{ and } \frac{a}{a'} = \frac{b}{b'}$$

expresses the same equality between products of the numbers:

$$a \times b' = a' \times b.$$

EXERCISES

329. If the full angle is taken for the unit of angular measure, find the measures of the angles containing 1°, $1'$, $1''$.

330. Prove that if $a : b = a' : b'$, then $(a + a') : (b + b') = a : b$.

331. Prove that if $a : a' = b : b' = c : c'$, then $(a+b) : c = (a'+b') : c'$.

332. Prove that if one side of a triangle is a common measure of the other two sides then the triangle is isosceles.

333. Prove that the perimeter and midline of a trapezoid circumscribed about a circle are commensurable.

334. Prove that the perimeter of an inscribed equilateral hexagon and the diameter of its circumscribed circle are commensurable.

335. In a triangle, find the greatest common measure of two segments: one between the orthocenter and barycenter, the other between the orthocenter and circumcenter.

336. Prove that the greatest common measure of two segments contains every their common measure a whole number of times.
Hint: All remainders in the Euclidean algorithm do.

337. Suppose that two given arcs on a given circle have the greatest common measure α. Show how to construct the arc α using only a compass. Consider the example where one of the given arcs contains 19°, and the other 360°.

338. Find the greatest common measure of two segments:
(a) one 1001 units long, the other 1105 units long;
(b) one 11,111, the other 1,111,111 units long.

339. Prove that the numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ are irrational.

340. Compute $\sqrt{5}$ with the precision of up to 0.0001.

341. Write $1/3$, $1/5$, $1/7$, $1/17$ as (finite or infinite) decimal fractions.

*342.** Prove that a rational number m/n is represented by a finite or repeating decimal fraction. Conversely, prove that a finite or repeating decimal fraction represents a rational number.

343. An acute angle of a parallelogram contains 60°, and its obtuse angle is divided by the diagonal in the proportion $3:1$. Find the ratio of the sides of the parallelogram.

*344.** Prove that the base of an isosceles triangle, whose angle at the vertex contains 36°, is incommensurable to the lateral side.
Hint: Draw the bisector from a vertex at the base, and compute angles of the two triangles thus formed.

2 Similarity of triangles

156. Preliminary remarks. In everyday life, we often encounter figures which have different sizes, but the same shape. Such figures are usually called **similar**. Thus, the same photographic picture printed in different sizes, or schemes of a building, or maps of a town, produced in different scales, provide examples of similar figures. Our concept of length of segments allows us to define precisely the concept of geometric **similarity of figures** and to describe ways of changing sizes of figures while preserving their shapes. Such changes of the size of a figure without changing its shape are called **similarity transformations**.

We begin our study of similar figures with the simplest case, namely similar triangles.

157. Homologous sides. We will need to consider triangles or polygons such that angles of one of them are respectively congruent to the angles of another. Let us agree to call **homologous** those sides of such triangles or polygons which are *adjacent* to the congruent angles (in triangles, such sides are also *opposite* to the congruent angles).

158. Definition. Two triangles are called **similar**, if: (1) the angles of one are respectively congruent to the angles of the other, and (2) the sides of one are proportional to the homologous sides of the other. Existence of such triangles is established by the following **lemma.** [2]

159. Lemma. ***A line*** (DE, Figure 166), ***parallel to any side*** (AC) ***of a given triangle*** (ABC), ***cuts off a triangle*** (DBE), ***similar to the given one.***

In a triangle ABC, let the line DE be parallel to the side AC. It is required to prove that the triangles DBE and ABC are similar. We will have to prove that (1) their angles are respectively congruent, and (2) their homologous sides are proportional.

(1) The angles of these triangles are respectively congruent, because $\angle B$ is their common angle, and $\angle D = \angle A$ and $\angle E = \angle C$ as corresponding angles between parallel lines (DE and AC), and a transversal (AB or CB respectively).

(2) Let us now prove that the sides of $\triangle DBE$ are proportional to the homologous sides of $\triangle ABC$, i.e. that

$$\frac{BD}{BA} = \frac{BE}{BC} = \frac{DE}{AC}.$$

For this, consider the following two cases.

(i) *The sides AB and DB have a common measure.* Divide the side AB into parts congruent to this common measure. Then DB will be divided into a *whole* number of such parts. Let the number of

[2] An auxiliary theorem introduced in order to facilitate the proof of another theorem which follows it is called a *lemma.*

such parts be m in DB and n in AB. From the division points, draw the set of lines parallel to AC, and another set of lines parallel to BC. Then BE and BC will be divided into congruent parts (§93), namely m in BE and n in BC. Likewise, DE will be divided into m congruent parts, and AC into n congruent parts, and moreover the parts of DE will be congruent to the parts of AC (as opposite sides of parallelograms). It becomes obvious now that

$$\frac{BD}{BA} = \frac{m}{n}, \quad \frac{BE}{BC} = \frac{m}{n}, \quad \frac{DE}{AC} = \frac{m}{n}.$$

Thus $BD : BA = BE : BC = DE : AC$.

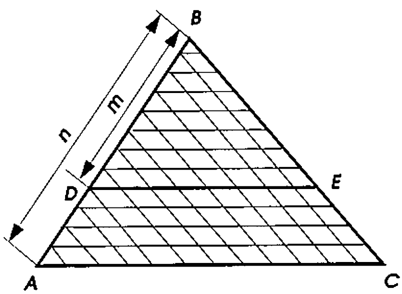

Figure 166

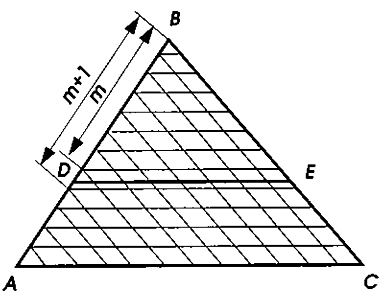

Figure 167

(ii) *The sides AB and DB do not have a common measure* (Figure 167). Approximate the values of each of the ratios $BD : BA$ and $BE : BC$ with the precision of up to $1/n$. For this, divide the side AB into n congruent parts, and through the division points, draw the set of lines parallel to AC. Then the side BC will also be divided into n congruent parts. Suppose that the $\frac{1}{n}$th part of AB is contained m times in DB with a remainder smaller than $\frac{1}{n}AB$. Then, as it is seen from Figure 167, the $\frac{1}{n}$th part of BC is contained in BE also m times with a remainder smaller than $\frac{1}{n}BC$. Similarly, drawing the set of lines parallel to BC, we find that the $\frac{1}{n}$th part of AC is contained in DE also m times with a remainder smaller than one such part. Therefore, with the precision of up to $\frac{1}{n}$th, we have

$$\frac{BD}{BA} \approx \frac{m}{n}, \quad \frac{BE}{BC} \approx \frac{m}{n}, \quad \frac{DE}{AC} \approx \frac{m}{n},$$

where we use the symbol "$\approx$" to express the approximate equality of numbers, which holds true within a required precision.

Taking first $n = 10$, then 100, then 1000, and so on, we find that the approximate values of the ratios computed with the same but arbitrary decimal precision, are equal to each other. Therefore the values of these ratios are expressed by the same infinite decimal fraction, and hence $BD : BA = BE : BC = DE : AC$.

160. Remarks. (1) The proven equalities can be written as the following three proportions:

$$\frac{BD}{BA} = \frac{BE}{BC}, \quad \frac{BE}{BC} = \frac{DE}{AC}, \quad \frac{DE}{AC} = \frac{BD}{BA}.$$

Transposing the mean terms we obtain:

$$\frac{BD}{BE} = \frac{BA}{BC}, \quad \frac{BE}{DE} = \frac{BC}{AC}, \quad \frac{DE}{BD} = \frac{AC}{BA}.$$

Thus, *if the sides of two triangles are proportional, then the ratio of any two sides of one triangle is equal to the ratio of the homologous sides of the other.*

(2) Similarity of figures is sometimes indicated by the sign $\sim$.

161. Three similarity tests for triangles.

Theorems. *If in two triangles,*

(1) ***two angles of one triangle are respectively congruent to two angles of the other,*** or

(2) ***two sides of one triangle are proportional to two sides of the other, and the angles between these sides are congruent,*** or

(3) ***if three sides of one triangle are proportional to three sides of the other,***

then such triangles are similar.

(1) Let ABC and $A'B'C'$ (Figure 168) be two triangles such that $\angle A = \angle A'$, $\angle B = \angle B'$, and therefore $\angle C = \angle C'$. It is required to prove that these triangles are similar.

Mark on AB the segment BD congruent to $A'B'$, and draw $DE \| AC$. Then we obtain auxiliary ΔDBE, which according to the lemma, is similar to ΔABC. On the other hand, ΔDBE is congruent to $\Delta A'B'C'$ by the ASA-test, because $BD = A'B'$ (by construction), $\angle B = \angle B'$ (by hypotheses), and $\angle D = \angle A'$ (since $\angle D = \angle A$ and $\angle A = \angle A'$). Clearly, if one of two congruent triangles is similar to another one, then the second one is also similar to it. Therefore $\Delta A'B'C' \sim \Delta ABC$.

(2) Let ABC and $A'B'C'$ (Figure 169) be two triangles such that $\angle B = \angle B'$, and $A'B' : AB = B'C' : BC$. It is required to prove that these triangles are similar.

As before, mark on AB the segment BD congruent to $A'B'$, and draw $DE \| AC$. Then we obtain auxiliary $\triangle DBE$ similar to $\triangle ABC$. Let us prove that it is congruent to $\triangle A'B'C'$. From the similarity of $\triangle DBE$ and $\triangle ABC$, it follows that $DB : AB = BE : BC$. Comparing this proportion with the given one, we note that the first ratios of both proportions coincide (since $DB = A'B'$), and hence the remaining ratios of these proportions are equal too. We see that $B'C' : BC = BE : BC$, i.e. that the segment $B'C'$ and BE have equal length when measured by the same unit BC, and hence $B'C' = BE$. We conclude now that the triangles DBE and $A'B'C'$ are congruent by the SAS-test, because they have congruent angles $\angle B$ and $\angle B'$ between respectively congruent sides. But $\triangle DBE$ is similar to $\triangle ABC$, and therefore $\triangle A'B'C'$ is also similar to $\triangle ABC$.

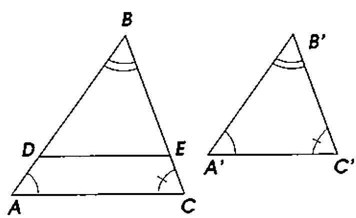

Figure 168

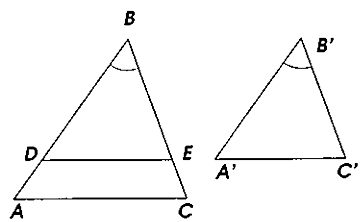

Figure 169

(3) Let ABC and $A'B'C'$ (Figure 169) be two triangles such that $A'B' : AB = B'C' : BC = A'C' : AC$. It is required to prove that these triangles are similar.

Repeating the same construction as before, let us show that $\triangle DBE$ and $\triangle A'B'C'$ are congruent. From the similarity of the triangles DBE and ABC, it follows that $DB : AB = BE : BC = DE : AC$. Comparing this series of ratios with the given one, we notice that the first ratios in both series are the same, and therefore all other ratios are also equal to each other. From $B'C' : BC = BE : BC$, we conclude that $B'C' = BE$, and from $A'C' : AC = DE : AC$ that $A'C' = DE$. We see now that the triangles DBE and $A'B'C'$ are congruent by the SSS-test, and since the first one of them is similar to $\triangle ABC$, then the second one is also similar to $\triangle ABC$.

162. Remarks (1) We would like to emphasize that the method applied in the proofs of the previous three theorems is the same. Namely, marking on a side of the greater triangle the segment congruent to the homologous side of the smaller triangle, and drawing the line parallel to another side, we form an auxiliary triangle similar to the greater given one. Then we apply the corresponding congruence test for triangles and derive from the hypotheses of the theorem and the similarity property that the auxiliary triangle is congruent to the smaller given one. Finally the conclusion about similarity of the given triangles is made.

(2) The three similarity tests are sometimes called the **AAA-test**, the **SAS-test**, and **SSS-test** respectively.

163. Similarity tests for right triangles. Since every two right angles are congruent, the following theorems follow directly from the AAA-test and SAS-test of similarity for general triangles and thus do not require separate proofs:

If in two right triangles,

(1) *an acute angle of one is congruent to an acute angle of the other,* or

(2) *legs of one are proportional to the legs of the other,*
then such right triangles are similar.

The following test does require a separate proof.

Theorem. ***If the hypotenuse and a leg of one right triangle are proportional to the hypotenuse and a leg of another one, then such triangles are similar.***

Let ABC and $A'B'C'$ be two triangles (Figure 170) such that the angles B and B' are right, and $A'B' : AB = A'C' : AC$. It is required to prove that these triangles are similar.

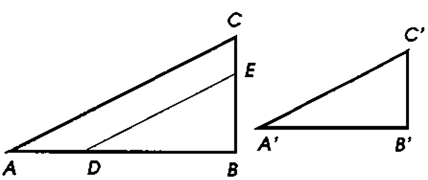

Figure 170

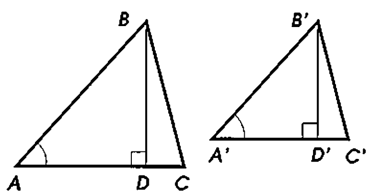

Figure 171

We apply the method used before. On the segment AB, mark $BD = A'B'$ and draw $DE \| AC$. Then we obtain the auxiliary triangle $\triangle DBE$ similar to $\triangle ABC$. Let us prove that it is congruent to $\triangle A'B'C'$. From the similarity of the triangles DBE and ABC, it follows that $DB : AB = DE : AC$. Comparing with the given proportion, we find that the first ratios in both proportions are the same, and therefore the second ratios are equal too, i.e. $DE : AC = A'C' : AC$, which shows that $DE = A'C'$. We see now that in the right triangles DBE and $A'B'C'$, the hypotenuses and one of the legs are respectively congruent. Thus the triangles are congruent, and since one of them is similar to $\triangle ABC$, then the other one is also similar to it.

164. Theorem. ***In similar triangles, homologous sides are proportional to*** **homologous altitudes**, i.e. to those altitudes which are dropped to the homologous sides.

Indeed, if triangles ABC and $A'B'C'$ (Figure 171) are similar, then the right triangles BAD and $B'A'D'$ are also similar (since $\angle A = \angle A'$), and therefore

$$\frac{BD}{B'D'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}.$$

EXERCISES

Prove theorems:

345. All equilateral triangles are similar.

346. All isosceles right triangles are similar.

347. Two isosceles triangles are similar if and only if their angles at the vertex are congruent.

348. In similar triangles, homologous sides are proportional to: (a) **homologous medians** (i.e. those medians which bisect homologous sides), and (b) **homologous bisectors** (i.e. the bisectors of respectively congruent angles).

349. Every segment parallel to the base of a triangle and connecting the other two sides is bisected by the median drawn from the vertex.

350. The line drawn through the midpoints of the bases of a trapezoid, passes through the intersection point of the other two sides, and through the intersection point of the diagonals.

351. A right triangle is divided by the altitude drawn to the hypotenuse into two triangles similar to it.

352. If a line divides a triangle into two similar triangles then these similar triangles are right.

353. Given three lines passing through the same point. If a point moves along one of the lines, then the ratio of the distances from this point to the other two lines remains fixed.

354. The line connecting the feet of two altitudes of any triangle cuts off a triangle similar to it. Derive from this that altitudes of any triangle are angle bisectors in another triangle, whose vertices are the feet of these altitudes.

*355.** If a median of a triangle cuts off a triangle similar to it, then the ratio of the homologous sides of these triangles is irrational.
Hint: Find this ratio.

Computation problems

356. In a trapezoid, the line parallel to the bases and passing through the intersection point of the diagonals is drawn. Compute the length of this line inside the trapezoid, if the bases are a units and b units long.

357. In a triangle ABC with sides a, b, and c units long, a line MN parallel to the side AC is drawn, cutting on the other two sides the segments $AM = BN$. Find the length of MN.

358. Into a right triangle with legs a and b units long, a square is inscribed in such a way that one of its angles is the right angle of the triangle, and the vertices of the square lie on the sides of the triangle. Find the perimeter of the square.

359. Two circles of radii R and r respectively are tangent externally at a point M. Compute the distance from M to the common external tangents of the circles.

3 Similarity of polygons

165. Definition. Two polygons with the same number of sides are called **similar**, if angles of one of them are respectively congruent to the angles of the other, and the homologous sides of these polygons are proportional. Thus, the polygon $ABCDE$ is similar to the polygon $A'B'C'D'E'$ (Figure 172), if

$$\angle A = \angle A',\ \angle B = \angle B',\ \angle C = \angle C',\ \angle D = \angle D',\ \angle E = \angle E',$$

and

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}.$$

Existence of such polygons is seen from the solution of the following problem.

166. Problem. *Given a polygon $ABCDE$, and a segment a, construct another polygon similar to the given one and such that its side homologous to the side AB is congruent to a* (Figure 173).

Here is a simple way to do this. On the side AB, mark $AB' = a$ (if $a > AB$, then the point B' lies on the extension of AB). Then draw all diagonals from the vertex A, and construct $B'C' \| BC$, $C'D' \| CD$ and $D'E' \| DE$. Then we obtain the polygon $AB'C'D'E'$ similar to the polygon $ABCDE$.

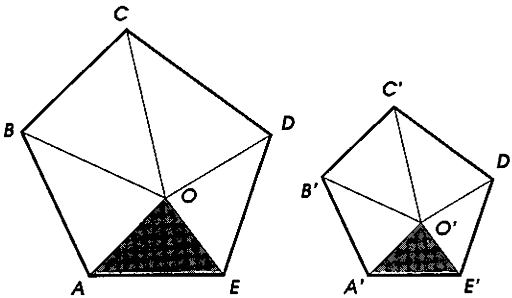

Figure 172

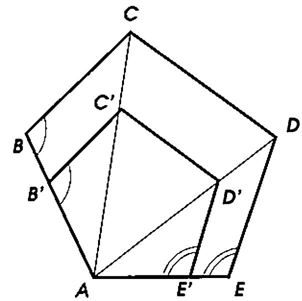

Figure 173

Indeed, firstly, the angles of one of them are congruent to the angles of the other: the angle A is common; $\angle B' = \angle B$ and $\angle E' = \angle E$ as corresponding angles between parallel lines and a transversal; $\angle C' = \angle C$ and $\angle D' = \angle D$, since these angles consist of parts respectively congruent to each other. Secondly, from similarity of triangles, we have the following proportions:

$$\text{from } \triangle AB'C' \sim \triangle ABC\text{:}\quad \frac{AB'}{AB} = \frac{B'C'}{BC} = \frac{AC'}{AC};$$

$$\text{from } \triangle AC'D' \sim \triangle ACD\text{:}\quad \frac{AC'}{AC} = \frac{C'D'}{CD} = \frac{AD'}{AD};$$

$$\text{from } \triangle AD'E' \sim \triangle ADE\text{:}\quad \frac{AD'}{AD} = \frac{D'E'}{DE} = \frac{AE'}{AE}.$$

Since the third ratio of the first row coincides with the first ratio of the second row, and the third ratio of the second row coincides with the first ratio of the third row, we conclude that all nine ratios are equal to each other. Discarding those of the ratios which involve the diagonals, we can write:

$$\frac{AB'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'E'}{DE} = \frac{AE'}{AE}.$$

We see therefore that in the polygons $ABCDE$ and $AB'C'D'E'$, which have the same number of vertices, the angles are respectively congruent, and the homologous sides are proportional. Thus these polygons are similar.

167. Remark. For triangles, as we have seen in §161, congruence of their angles implies proportionality of their sides, and conversely, proportionality of the sides implies congruence of the angles. As a result, congruence of angles alone, or proportionality of sides alone is a sufficient test of similarity of triangles. For polygons however, congruence of angles alone, or proportionality of sides alone is insufficient to claim similarity. For example, a square and a rectangle have congruent angles, but non-proportional sides, and a square and a rhombus have proportional sides, but non-congruent angles.

168. Theorem. ***Similar polygons can be partitioned into an equal number of respectively similar triangles positioned in the same way.***

For instance, similar polygons $ABCDE$ and $AB'C'D'E'$ (Figure 173) are divided by the diagonals into similar triangles which are positioned in the same way. Obviously, this method applies to every convex polygon. Let us point out another way which also works for convex polygons.

Inside the polygon $ABCDE$ (Figure 172), take any point O and connect it to all the vertices. Then the polygon $ABCDE$ will be partitioned into as many triangles as it has sides. Pick one of them, say, $\triangle AOE$ (it is shaded on the Figure 172), and on the homologous side $A'E'$ of the other polygon, construct the angles $O'A'E'$ and $O'E'A'$ respectively congruent to the angles OAE and OEA. Connect the intersection point O' with the remaining vertices of the polygon $A'B'C'D'E'$. Then this polygon will be partitioned into the same number of triangles. Let us prove that the triangles of the first polygon are respectively similar to the triangles of the second one.

Indeed, $\triangle AOE$ is similar to $\triangle A'O'E'$ by construction. To prove similarity of the adjacent triangles AOB and $A'O'B'$, we take into account that similarity of the polygons implies that

$$\angle BAE = \angle B'A'E', \text{ and } \frac{BA}{B'A'} = \frac{AE}{A'E'},$$

and similarity of the triangles AOE and $A'O'E'$ implies that

$$\angle OAE = \angle O'A'E', \text{ and } \frac{AO}{A'O'} = \frac{AE}{A'E'}.$$

It follows that

$$\angle BAO = \angle B'A'O', \text{ and } \frac{BA}{B'A'} = \frac{AO}{A'O'}.$$

We see that the triangles AOB and $A'O'B'$ have congruent angles contained between two proportional sides, and are therefore similar.

In exactly the same way, we then prove similarity of $\triangle BOC$ and $\triangle B'O'C'$, then of $\triangle COD$ and $\triangle C'O'D'$, etc. Obviously, the similar triangles are positioned in their respective polygons in the same way.

In order to prove the theorem for non-convex polygons, it suffices to partition them in the same way into convex ones, by the method explained in §82 (see Remark (2)).

169. Theorem. ***Perimeters of similar polygons are proportional to homologous sides.***

Indeed, if polygons $ABCDE$ and $A'B'C'D'E'$ (Figure 172) are similar, then by definition

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'} = k,$$

where k is some real number. This means that $AB = k(A'B')$, $BC = k(B'C')$, etc. Adding up, we find

$$AB+BC+CD+DE+EA = k(A'B'+B'C'+C'D'+D'E'+E'A'),$$

and hence

$$\frac{AB+BC+CD+DE+EA}{A'B'+B'C'+C'D'+D'E'+E'A'} = k.$$

Remark. This is a general property of proportions: given a row of equal ratios, the sum of the first terms of the ratios are to the sum of the second terms, as each of the first terms is to the corresponding second term.

EXERCISES

360. Prove that all squares are similar.

361. Prove that two rectangles are similar if and only if they have equal ratios of non-parallel sides.

362. Prove that two rhombi are similar if and only if they have congruent angles.

363. How does the previous result change if the rhombi are replaced by arbitrary equilateral polygons?

364. Prove that two kites are similar if and only if the angles of one of them are respectively congruent to the angles of the other.

365. Prove that two inscribed quadrilaterals with perpendicular diagonals are similar if and only if they have respectively congruent angles.

366.* How does the previous result change, if the diagonals of the inscribed quadrilaterals form congruent angles, other than d?

367. Prove that two circumscribed quadrilaterals are similar if and only if the angles of one of them are respectively congruent to the angles of the other.

368. How does the previous result change if quadrilaterals are replaced by arbitrary polygons?

369. Two quadrilaterals are cut into two congruent equilateral triangles each. Prove that the quadrilaterals are similar.

370. How does the previous result change if the equilateral triangles are replaced with right isosceles triangles?

4 Proportionality theorems

170. Thales' theorem. The following result was known to the Greek philosopher *Thales of Miletus* (624 B.C. – 547 B.C.)

Theorem. ***The sides of an angle*** (ABC, Figure 174) ***intersected by a series of parallel lines*** (DD', EE', FF', ...) ***are divided by them into proportional parts.***

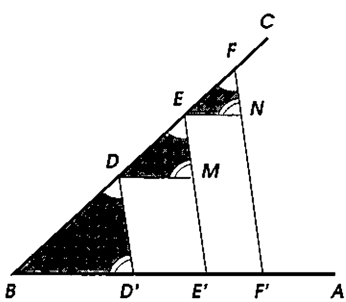

Figure 174

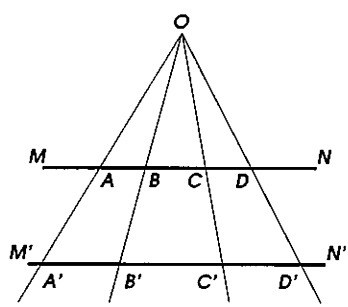

Figure 175

It is required to prove that

$$\frac{BD}{BD'} = \frac{DE}{D'E'} = \frac{EF}{E'F'} = \dots,$$

or, equivalently, that

$$\frac{BD}{DE} = \frac{BD'}{D'E'}, \quad \frac{DE}{EF} = \frac{D'E'}{E'F'}, \ \dots$$

Draw the auxiliary lines DM, EN, ..., parallel to BA. We obtain the triangles BDD', DEM, EFN, ..., which are all similar to each other, since their angles are respectively congruent (due to the property of parallel lines intersected by a transversal). It follows from the similarity that

$$\frac{BD}{BD'} = \frac{DE}{DM} = \frac{EF}{EN} = \dots$$

Replacing in this sequence of equal ratios the segments: DM with $D'E'$, EN with $E'F'$, ..., (congruent to them as opposite sides of parallelograms), we obtain what was required to prove.

171. Theorem. ***Two parallel lines*** (MN and $M'N'$, Figure 175) ***intersected by a series of lines*** (OA, OB, OC, ...), ***drawn from the same point*** (O), ***are divided by these lines into proportional parts.***

It is required to prove that the segments AB, BC, CD, ... of the line MN are proportional to the segments $A'B'$, $B'C'$, $C'D'$, ... of the line $M'N'$.

From the similarity of triangles (§159): $OAB \sim OA'B'$ and $OBC \sim OB'C'$, we derive:

$$\frac{AB}{A'B'} = \frac{BO}{B'O} \text{ and } \frac{BO}{B'O} = \frac{BC}{B'C'},$$

and conclude that $AB : A'B' = BC' : B'C'$. The proportionality of the other segments is proved similarly.

172. Problem. *To divide a line segment* AB (Figure 176) *into three parts in the proportion* $m : n : p$, *where* m, n, *and* p *are given segments or given whole numbers.*

Issue a ray AC making an arbitrary angle with AB, and mark on it, starting from the point A, the segments congruent to the given segments m, n, and p. Connect the endpoint F of the segment p with B, and through the endpoints G and H of the marked segments, draw the lines GD and HE parallel to FB. Then the segment AB will be divided by the points D and E in the proportion $m : n : p$.

When m, n, and p denote given whole numbers, e.g. $2, 5, 3$, then the construction is performed similarly, except that the segments marked on AC are to have lengths 2, 5, and 3 in the same arbitrary units.

The described construction applies, of course, to division of segments into any number of parts.

173. `Problem.` *Given three segments a, b, and c, find a fourth segment to form a proportion* (Figure 177), i.e. find a segment x such that $a : b = c : x$.

On the sides of an arbitrary angle ABC, mark the segments $BD = a$, $BF = b$, $DE = c$. Connect D and F, and construct $EG \| DF$. The required segment is FG.

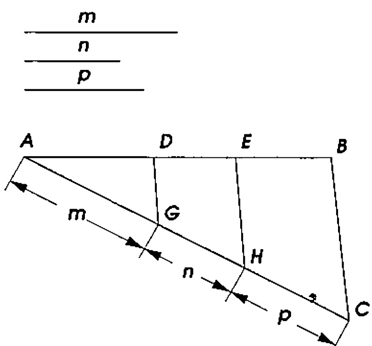

Figure 176

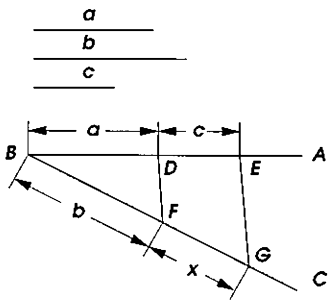

Figure 177

174. A property of bisectors.

`Theorem.` ***The bisector*** (BD, Figure 178) ***of any angle of a triangle*** (ABC) ***divides the opposite side into parts*** (AD and DC) ***proportional to the adjacent sides.***

It is required to prove that if $\angle ABD = \angle DBC$, then

$$\frac{AD}{DC} = \frac{AB}{BC}.$$

Draw CE parallel to BD up to the intersection at a point E with the extension of the side AB. Then, according to Thales' theorem (§170), we will have the proportion $AD : DC = AB : BE$. To derive from this the required proportion, it suffices to show that $BE = BC$, i.e. that $\triangle CBE$ is isosceles. In this triangle, $\angle E = \angle ABD$ and $\angle BCE = \angle DBC$ (respectively as corresponding and as alternate angles formed by a transversal with parallel lines). But $\angle ABD = \angle DBC$ by the hypothesis, hence $\angle E = \angle BCE$, and therefore BC and BE are congruent as the sides opposite to congruent angles.

Example. Let $AB = 30$, $BC = 24$, and $AC = 36$ *cm*. We can denote AD by the letter x and write the proportion:

$$\frac{x}{36 - x} = \frac{30}{24}, \quad \text{i.e.} \quad \frac{x}{36 - x} = \frac{5}{4}.$$

We find therefore: $4x = 180 - 5x$, or $9x = 180$, i.e. $x = 20$. Thus $AD = 20$ *cm*, and $DC = 36 - x = 16$ *cm*.

175. Theorem. ***The bisector*** (BD, Figure 179) ***of an exterior angle*** (CBF) ***at the vertex of a triangle*** (ABC) ***intersects the extension of the base*** (AC) ***at a point*** (D) ***such that the distances*** (DA and DC) ***from this point to the endpoints of the base are proportional to the lateral sides*** (AB and BC) ***of the triangle.***

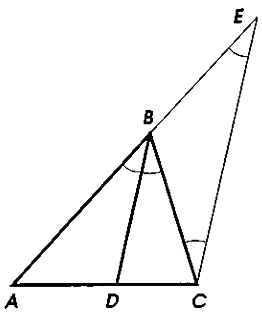

Figure 178

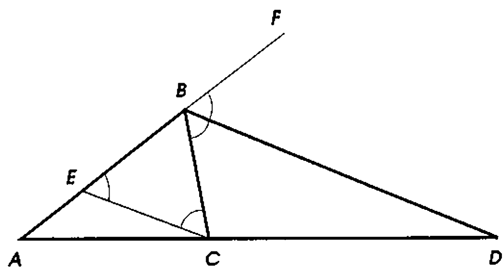

Figure 179

In other words, it is required to prove that if $\angle CBD = \angle FBD$, then

$$\frac{DA}{DC} = \frac{AB}{BC}.$$

Drawing $CE \| BD$, we can write the proportion: $DA : DC = BA : BE$. Since $\angle BEC = \angle FBD$ and $\angle BCE = \angle CBD$ (respectively as corresponding and as alternate angles formed by parallel lines with a transversal), and $\angle FBD = \angle CBD$ by the hypothesis, we have $\angle BEC = \angle BCE$. Therefore ΔEBC is isosceles, i.e. $BE = BC$. Replacing, in the proportion we already have, the segment BC with the congruent segment BE, we obtain the required proportion: $DA : DC = BA : BC$.

Remark. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base. This is an exceptional case in the formulation of the theorem and in its proof.

EXERCISES

371. Prove that if proportional segments are marked on the sides of an angle starting from the vertex, then the lines connecting their endpoints are parallel.

372. Construct a line segment connecting lateral sides of a given trapezoid and parallel to its bases, such that it is divided by the diagonals into three congruent parts.

373. Construct a triangle, given the angle at the vertex, the base, and its ratio to one of the lateral sides.

374. Prove that the bisector of the angle between two non-congruent sides of a triangle is smaller than the median drawn from the same vertex.

375. In a triangle with sides 12, 15, and 18 *cm*, a circle is drawn tangent to both smaller sides and with the center lying on the greatest side. Find the segments into which the center divides the greatest side.

376. Through a given point on the bisector of a given angle, draw a line whose part inside the angle is divided by the point in the given proportion $m : n$.

377. Construct a triangle, given the angle at the vertex, the base, and the point on the base where it meets the angle bisector.

378. Into a given circle, inscribe a triangle, given its base and the ratio of the other two sides.

*379.** Construct a triangle, given two of its sides and the bisector of the angle between them.
`Hint:` Examine Figure 178, and construct $\triangle CBE$ first.

*380.** In $\triangle ABC$, the side $AC = 6$ *cm*, $BC = 4$ *cm*, and $\angle B = 2\angle A$. Compute AB.
`Hint:` See Example in §174.

381. Given two points A and B on an infinite line, find a third point C on this line, such that $CA : CB = m : n$, where m and n are given segments or given numbers. (If $m \neq n$ there are two such points: one between A and B, the other outside the segment AB.)

*382.** Given two points A and B, find the geometric locus of points M such that MA and MB have a given ratio $m : n$.
`Hint:` The answer is often called **Apollonius' circle** after the Greek geometer *Apollonius of Perga* (262 – 190 B.C.)

*383.** Into a given circle, inscribe a triangle, given its base, and the ratio of the median, bisecting the base, to one of the lateral sides.

5 Homothety

176. Homothetic figures. Suppose we are given (see Figure 180): a figure Φ, a point S, which we will call the **center of homothety**, and a positive number k, which we will call the **similarity coefficient** (or **homothety coefficient**). Take an arbitrary point A in the figure Φ and draw through it the ray SA drawn from the center S. Find on this ray the point A' such that the ratio $SA' : SA$ is equal to k. Thus, if $K < 1$, e.g. $k = 1/2$, then the point A' lies between S and A (as in Figure 180), and if $k > 1$, e.g. $k = 3/2$, then the point A' lies beyond the segment SA. Take another point B of the figure Φ, and repeat the same construction as we explained for A, i.e. on the ray SB, find the point B' such that $SB' : SB = k$. Imagine now that, keeping the point S and the number k unchanged, we find for every point of the figure Φ the corresponding new point obtained by the same construction. Then the geometric locus of all such points is a new figure Φ'. The resulting figure Φ' is called **homothetic** to the figure Φ with respect to the center S and with the given coefficient k. The transformation of the figure Φ into Φ' is called a **homothety**, or **similarity transformation**, with the center S and coefficient k.

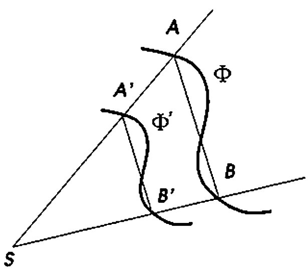

Figure 180

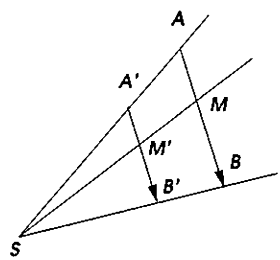

Figure 181

177. Theorem. ***A figure homothetic to a line segment*** (AB, Figure 181) ***is a line segment*** ($A'B'$), ***parallel to the first one and such that the ratio of this segment to the first one is equal to the homothety coefficient.***

Find points A' and B' homothetic to the endpoints A and B of the first segment with respect to the given center S and with the given homothety coefficient k. The points A' and B' lie on the rays SA and SB respectively, and $SA' : SA = k = SB' : SB$. Connect A' with B' and prove that $A'B' \| AB$, and $A'B' : AB = k$. Indeed, $\triangle A'SB' \sim \triangle ASB$ since they have the common angle S, and their sides containing this angle are proportional. From the similarity of these triangles, it follows that $A'B' : AB = SA' : SA = k$, and that $\angle BAS = \angle B'A'S$, and hence that $A'B' \| AB$.

Let us prove now that the segment $A'B'$ is the figure homothetic to AB. For this, pick any point M on AB and draw the ray SM. Let M' be the point where this ray intersects the line $A'B'$. The triangles $M'A'S$ and MAS are similar because the angles of one of them are congruent to the angles of the other. Therefore $SA' : SM = SA' : SA = k$, i.e. M' is the point homothetic to M with respect to the center S and with the coefficient k. Thus, for any point on AB, the point homothetic to it lies on $A'B'$. *Vice versa*, picking any point M' on $A'B'$ and intersecting the ray SM' with AB, we similarly find that M' is homothetic to M. Thus the segment $A'B'$ is the figure homothetic to AB.

Remark. Note that the segment $A'B'$ with the endpoints respectively homothetic to the endpoints of the segment AB, is not only parallel to AB, but also has the same direction (indicated in Figure 181 by arrows).

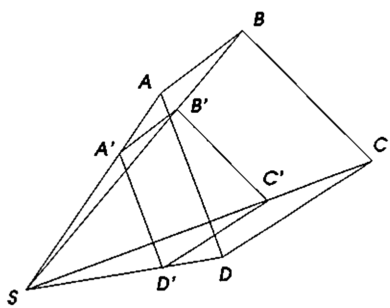

Figure 182

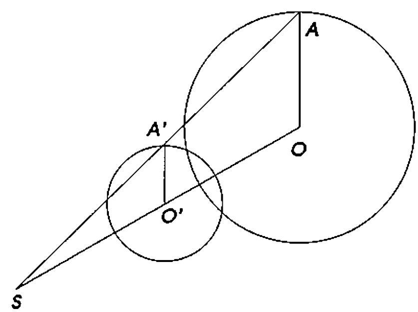

Figure 183

178. Theorem. ***The figure homothetic to a polygon*** $(ABCD$, Figure 182) ***is a polygon*** $(A'B'C'D')$ ***similar to the first one, and such that its sides are parallel to the homologous sides of the first polygon, and the ratio of the homologous sides is equal to the homothety coefficient*** (k).

Indeed, according to the previous theorem, the figure homothetic to a polygon $ABCD$ is formed by the segments parallel to its sides, directed the same way, and proportional to them with the proportionality coefficient k. Therefore the figure is a polygon $A'B'C'D'$, whose angles are respectively congruent to the angles of $ABCD$ (as angles with parallel respective sides, §79), and whose homologous sides are proportional to the sides of $ABCD$. Thus these polygons are similar.

Remark. One can define similarity of arbitrary geometric figures as follows: two figures are called **similar** if one of them is congruent to a figure homothetic to the other. Thus, homothetic figures are similar in this sense. The theorem shows that our earlier definition of similar polygons (§165) agrees with the general definition of similar figures.

179. Theorem. ***The figure homothetic to a circle*** (centered at O, Figure 183), ***is a circle such that the ratio of its radius to the radius of the first circle is equal to the homothety coefficient, and whose center*** (O') ***is the point homothetic to the center of the first circle.***

Let S be the center of homothety, and k the coefficient. Pick an arbitrary radius OA of the given circle and construct the segment $O'A'$ homothetic to it. Then $O'A' : OA = k$ by the result of §177, i.e. $O'A' = k\,OA$. When the radius OA rotates about the center O, the length of the segments $O'A'$ remains therefore constant, and the point O' homothetic to the fixed point O, remains fixed. Thus the point A' describes the circle with the center O' and the radius congruent to k times the radius of the given circle.

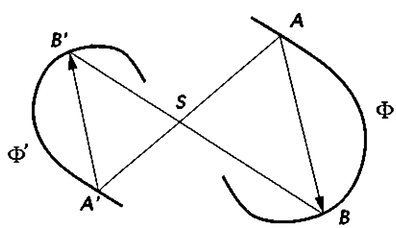

Figure 184

180. Negative homothety coefficients. Suppose we are given a figure Φ, a point S, and a positive number k. We can alter the construction of the figure homothetic to Φ in the following fashion. Pick a point A (Figure 184) of the figure Φ, issue from S the ray SA, and extend it beyond the point S. On the extension of this ray, mark the point A' such that $SA' : SA = k$. When this construction is repeated (keeping S and k the same) for all points A of the figure Φ, the locus of the corresponding points A' is a new figure Φ'. The figure Φ' is also considered homothetic to the figure Φ with respect to the center S, but with the **negative homothety** coefficient equal to $-k$.

We suggest that the reader verifies the following facts about homotheties with negative coefficients:

(1) *The figure homothetic with a negative coefficient* $-k$ *to a line segment* AB (Figure 184) *is a line segment* $A'B'$ *parallel to* AB, *congruent to* $k\,AB$, *and having the direction opposite to the direction of* AB.

(2) *The similarity transformation with the center* S *and coefficient* -1 *is the same as the central symmetry about the center* S.

(3) *Two figures, homothetic to a given figure about a center* S *and with coefficients* k *and* $-k$ *respectively, are centrally symmetric to each other about the center* S.

(4) *On the number line* (§163), *the points representing the numbers* k *and* $-k$ *are homothetic to the point representing the number* 1 *with respect to the center* 0, *and with the homothety coefficients equal to* k *and* $-k$ *respectively.*

181. The method of homothety. This method can be successfully applied to solving many construction problems. The idea is to construct first a figure *similar* to the required one, and then to obtain the required figure by means of a similarity transformation. The homothety method is particularly convenient when only one of the given quantities is a length, and all others are angles or ratios, such as in the problems: to construct a triangle, given its angle, side, and the ratio of the other two sides, or given two angles and a certain segment (an altitude, median, angle bisector, etc.); to construct a square, given the sum or the difference of its side and the diagonal. Let us solve, for example, the following problem.

`Problem 1.` *To construct a triangle* ABC, *given the angle* C, *the ratio of its sides* $AC : BC$, *and the altitude* h, *dropped from the vertex of this angle to the opposite side* (Figure 185).

Let $AC : BC = m : n$, where m and n are two given segments or two given numbers. Construct the angle C, and on its sides, mark the segments CA' and CB', proportional to m and n. When m and n are segments, we may take $CA' = m$ and $CB' = n$. If m and n are whole numbers, then picking an arbitrary segment l, we may construct $CA' = ml$ and $CB' = nl$. In both cases, we have $CA' : CB' = m : n$.

The triangle $A'B'C$ is, evidently, similar to the required one. To obtain the required triangle, construct the altitude CD' of the triangle $A'B'C$ and denote it h'. Now pick an arbitrary homothety center and construct the triangle homothetic to the triangle $A'B'C$ with the homothety coefficient equal to h/h'. The resulting triangle will be the required one.

It is most convenient to pick the center at the point C. Then the construction becomes especially simple (Figure 185). Extend the altitude CD' of the triangle $A'B'C$, mark on it the segment CD congruent to h, and draw through its endpoint D the line AB parallel to $A'B'$. The triangle ABC is the required one.

The position of the required figure in problems of this kind remains arbitrary. In some other problems, it is required to construct a figure in a quite definite position with respect to given points and lines. It can happen, that discarding one of these requirements, we obtain infinitely many solutions *similar* to the required figure. Then the method of homothety becomes useful. Here are some examples.

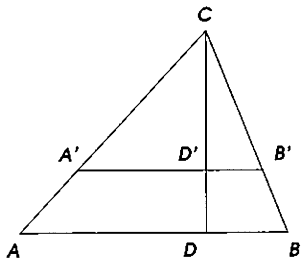

Figure 185

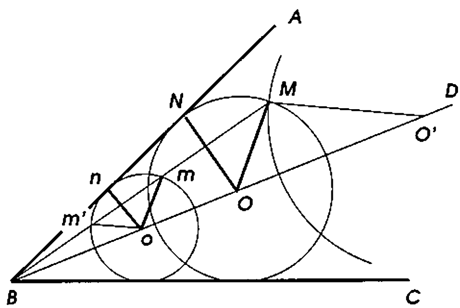

Figure 186

182. Problem 2. *Into a given angle ABC, to inscribe a circle that would pass through a given point M* (Figure 186).

Discard temporarily the requirement for the circle to pass through the point M. The remaining condition is satisfied by infinitely many circles whose centers lie on the bisector BD of the given angle. Construct one such circle, e.g. the one with the center at some point o. Take on it the point m *homothetic* with respect to the center B to the point M, i.e. lying on the ray BM, and draw the radius mo. If we now construct $MO \| mo$, then the point O will be the center of the required circle.

Indeed, draw the perpendiculars ON and on to the side AB. We obtain similar triangles: $MBO \sim mBo$, and $NBO \sim nBo$. From their similarity, we have: $MO : mo = BO : Bo$ and $NO : no = BO : Bo$, and therefore $MO : mo = NO : no$. But $mo = no$, and hence $MO = NO$, i.e. the circle described by the radius OM about the center O is tangent to the side AB. Since its center lies on the bisector of the angle, it is tangent to the side BC as well.

If instead of the point m on the auxiliary circle, the other intersection point m' of this circle with the ray BM is taken as homothetic to M, then another center O' of the required circle will be constructed. Thus the problem admits two solutions.

183. `Problem 3.` *Into a given triangle ABC, to inscribe a rhombus with a given acute angle, in such a way that one of its sides lies on the base AB of the triangle, and two vertices on the lateral sides AC and BC* (Figure 187).

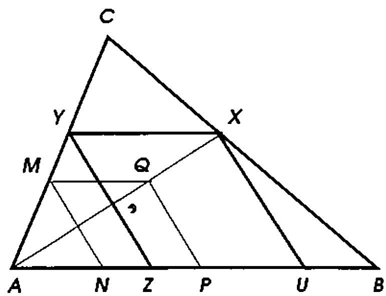

Figure 187

Discard temporarily the requirement for one of the vertices to lie on the side BC. Then there are infinitely many rhombi satisfying the remaining conditions. Construct one of them. For this, take on the side AC an arbitrary point M and construct the angle, congruent to the given one, with the vertex at the point M, and such that one of its sides is parallel to the base AB and the other intersects the base at some point N. On the side AB, mark a segment NP congruent to MN, and construct the rhombus with the sides MN and NP.

Let Q be the fourth vertex of this rhombus. Taking A for the center of homothety, construct the rhombus homothetic to the rhombus $MNPQ$, and choose the homothety coefficient such that the vertex of the new rhombus corresponding to the vertex Q turns out to lie on the side BC of the triangle. For this, extend the ray AQ up to its intersection with the side BC at some point X. This point will be one of the vertices of the required rhombus. Drawing through X the lines parallel to the sides of the rhombus $MNPQ$, we obtain the required rhombus $XYZU$.

EXERCISES

Prove theorems:

384. If the radii of two circles rotate remaining parallel to each other, then the lines passing through the endpoints of such radii intersect the line of centers at a fixed point.

385. Two circles on the plane are homothetic to each other with respect to a suitable center (even two centers, for one the homothety coefficient is negative, and for the other positive).
Hint: The centers of homothety are the fixed intersection points from the previous problem.

Find the geometric locus of:

386. Midpoints of all chords passing through a given point on a circle.

387. Points dividing all chords passing through a given point on a circle in a fixed ratio $m : n$.

388. Points from which the distances to the sides of a given angle have a fixed ratio.

Construction problems

389. Through a point given in the interior of an angle, draw a line such that its segments between the point and the sides of the angle have a given ratio $m : n$.

390. About a given square, circumscribe a triangle similar to a given one.

391. Find a point inside a triangle such that the three perpendiculars dropped from this point to the sides of the triangle are in the given proportion $m : n : p$.

392. Construct a triangle, given the angle at the vertex, the altitude, and the ratio in which its foot divides the base.

393. Construct a triangle, given its angles, and the sum or the difference of the base and the altitude.

394. Construct an isosceles triangle, given the angle at the vertex, and the sum of the base with the altitude.

395. Construct a triangle, given its angles and the radius of its circumscribed circle.

396. Given $\angle AOB$ and a point C in its interior. On the side OB, find a point M equidistant from OA and C.

397. Construct a triangle, given the ratio of its altitude to the base, the angle at the vertex, and the median drawn to one of its lateral sides

398. Into a given disk segment, inscribe a square such that one of its sides lies on the chord, and the opposite vertices on the arc.

399. Into a given triangle, inscribe a rectangle with the given ratio of the sides $m : n$, so that one of its sides lies on the base of the triangle, and the opposite vertices on the lateral sides.

6 Geometric mean

184. Definition. The **geometric mean** between two segments a and c is defined to be a third segment b such that $a : b = b : c$. More generally, the same definition applies to any quantities of the same denomination. When a, b, and c are positive numbers, the relationship $a : b = b : c$ can be rewritten as

$$b^2 = ac, \text{ or } b = \sqrt{ac}.$$

185. Theorem. ***In a right triangle:***

(1) ***the altitude dropped from the vertex of the right angle is the geometric mean between two segments into which the foot of the altitude divides the hypotenuse,*** and

(2) ***each leg is the geometric mean between the hypotenuse and the segment of it which is adjacent to the leg.***

Let AD (Figure 188) be the altitude dropped from the vertex of the right angle A to the hypotenuse BC. It is required to prove the following proportions:

$$(1)\ \frac{BD}{AD} = \frac{AD}{DC}, \quad (2)\ \frac{BC}{AB} = \frac{AB}{BD} \text{ and } \frac{BC}{AC} = \frac{AC}{DC}.$$

The first proportion is derived from similarity of the triangles BDA and ADC. These triangles are similar because

$$\angle 1 = \angle 4 \text{ and } \angle 2 = \angle 3$$

as angles with perpendicular respective sides (§80). The sides BD and AD of $\triangle BDA$ form the first ratio of the required proportion.

The homologous sides of $\triangle ADC$ are AD and DC, [3] and therefore $BD : AD = AD : DC$.

The second proportion is derived from similarity of the triangles ABC and BDA. These triangles are similar because both are right, and $\angle B$ is their common acute angle. The sides BC and AB of $\triangle ABC$ form the first ratio of the required proportion. The homologous sides of $\triangle BDA$ are AB and BD, and therefore $BC : AB = AB : BD$.

The last proportion is derived in the same manner from the similarity of the triangles ABC and ADC.

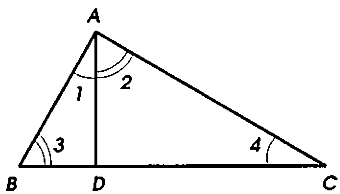

Figure 188

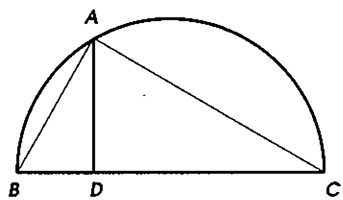

Figure 189

186. Corollary. Let A (Figure 189) be any point on a circle, described about a diameter BC. Connecting this point by chords with the endpoints of the diameter we obtain a right triangle such that its hypotenuse is the diameter, and its legs are the chords. Applying the theorem to this triangle we arrive at the following conclusion:

The perpendicular dropped from any point of a circle to its diameter is the geometric mean between the segments into which the foot of the perpendicular divides the diameter, and the chord connecting this point with an endpoint of the diameter is the geometric mean between the diameter and the segment of it adjacent to the chord.

187. Problem. *To construct the geometric mean between two segments a and c.*

We give two solutions.

(1) On a line (Figure 190), mark segments $AB = a$ and $BC = c$ next to each other, and describe a semicircle on AC as the diameter.

---

[3]In order to avoid mistakes in determining which sides of similar triangles are homologous to each other, it is convenient to mark angles opposite to the sides in question of one triangle, then find the angles congruent to them in the other triangle, and then take the sides opposite to these angles. For instance, the sides BD and AD of $\triangle BDA$ are opposite to the angles 1 and 3; these angles are congruent to the angles 4 and 2 of $\triangle ADC$, which are opposite to the sides AD and DC. Thus the sides AD and DC correspond to BD and AD respectively.

From the point B, erect the perpendicular to AC up to the intersection point D with the semicircle. The perpendicular BD is the required geometric mean between AB and BC.

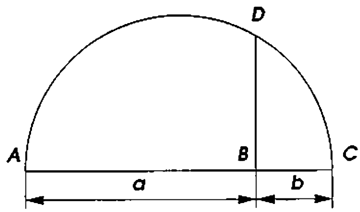

Figure 190

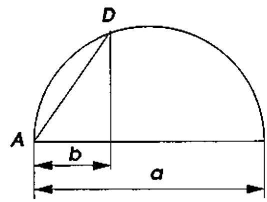

Figure 191

(2) From the endpoint A of a ray (Figure 191), mark the given segments a and b. On the greater of them, describe a semicircle. From the endpoint of the smaller one, erect the perpendicular up to the intersection point D with the semicircle, and connect D with A. The chord AD is the required geometric mean between a and b.

188. The Pythagorean Theorem. The previous theorems allow one to obtain a remarkable relationship between the sides of any right triangle. This relationship was proved by the Greek geometer *Pythagoras of Samos* (who lived from about 570 B.C.to about 475 B.C.) and is named after him.

Theorem. ***If the sides of a right triangle are measured with the same unit, then the square of the length of its hypotenuse is equal to the sum of the squares of the lengths of its legs.***

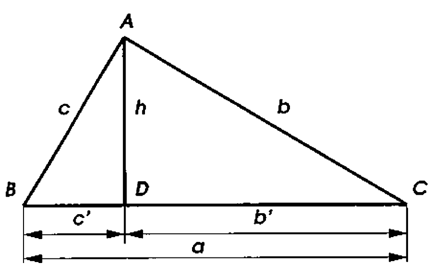

Figure 192

Let ABC (Figure 192) be a right triangle, and AD the altitude dropped to the hypotenuse from the vertex of the right angle. Suppose that the sides and the segments of the hypotenuse are measured by the same unit, and their lengths are expressed by the numbers a, b, c, c' and b'.[4] Applying the theorem of §185, we obtain the proportions:

$$a : c = c : c' \text{ and } a : b = b : b',$$

or equivalently:

$$ac' = c^2 \quad \text{and} \quad ab' = b^2.$$

Adding these equalities, we find:

$$ac' + ab' = c^2 + b^2, \quad \text{or} \quad a(c' + b') = c^2 + b^2.$$

But $c' + b' = a$, and therefore $a^2 = b^2 + c^2$.

This theorem is often stated in short: *the square of the hypotenuse equals the sum of the squares of the legs.*

Example. Suppose that the legs measured with some linear unit are expressed by the numbers 3 and 4. Then the hypotenuse is expressed in the same units by a number x such that

$$x^2 = 3^2 + 4^2 = 9 + 16 = 25, \text{ and hence } x = \sqrt{25} = 5.$$

Remark. The right triangle with the sides 3, 4, and 5 is sometimes called *Egyptian* because it was known to ancient Egyptians. It is believed they were using this triangle to construct right angles on the land surface in the following way. A circular rope marked by 12 knots spaced equally would be stretched around three poles to form a triangle with the sides of 3, 4, and 5 spacings. Then the angle between the sides equal to 3 and 4 would turn out to be right. [5]

Yet another formulation of the Pythagorean theorem, namely the one known to Pythagoras himself, will be given in §259.

189. Corollary. *The squares of the legs have the same ratio as the segments of the hypotenuse adjacent to them.*

Indeed, from formulas in §188 we find $c^2 : b^2 = ac' : ab' = c' : b'$.

Remarks. (1) The three equalities

$$ac' = c^2, \ ab' = b^2, \ a^2 = b^2 + c^2,$$

[4] It is customary to denote sides of triangles by the lowercase letters corresponding to the uppercase letters which label the opposite vertices.

[5] Right triangles whose sides are measured by *whole* numbers are called **Pythagorean**. One can prove that the legs x and y, and the hypotenuse z of such triangles are expressed by the formulas: $x = 2ab, y = a^2 - b^2, z = a^2 + b^2$, where a and b are arbitrary whole numbers such that $a > b$.

can be supplemented by two more:

$$b' + c' = a, \quad \text{and} \quad h^2 = b'c',$$

where h denotes the length of the altitude AD (Figure 192). The third of the equalities, as we have seen, is a consequence of the first two and of the fourth, so that only four of the five equalities are independent. As a result, given two of the six numbers a, b, c, b', c' and h, we can compute the remaining four. For example, suppose we are given the segments of the hypotenuse $b' = 5$ and $c' = 7$. Then

$$a = b' + c' = 12, \ c = \sqrt{ac'} = \sqrt{12 \cdot 7} = \sqrt{84} = 2\sqrt{21},$$

$$b = \sqrt{ab'} = \sqrt{12 \cdot 5} = \sqrt{60}, \ h = \sqrt{b'c'} = \sqrt{5 \cdot 7} = \sqrt{35}.$$

(2) Later on we will often say: "the square of a segment" instead of "the square of the number expressing the length of the segment," or "the product of segments" instead of "the product of numbers expressing the lengths of the segments." We will assume therefore that all segments have been measured using the same unit of length.

190. Theorem. ***In every triangle, the square of a side opposite to an acute angle is equal to the sum of the squares of the two other sides minus twice the product of (any) one of these two sides and the segment of this side between the vertex of the acute angle and the foot of the altitude drawn to this side.***

Let BC be the side of $\triangle ABC$ (Figures 193 and 194), opposite to the acute angle A, and BD the altitude dropped to another side, e.g. AC, (or to its extension). It is required to prove that

$$BC^2 = AB^2 + AC^2 - 2AC \cdot AD,$$

or, using the notation of the segments by single lowercase letters as shown on Figures 193 or 194, that

$$a^2 = b^2 + c^2 - 2bc'.$$

From the right triangle BDC, we have:

$$a^2 = h^2 + (a')^2. \qquad (*)$$

Let us compute each of the squares h^2 and $(a')^2$. From the right triangle BAD, we find: $h^2 = c^2 - (c')^2$. On the other hand, $a' = b - c'$

(Figure 193) or $a' = c' - b$ (Figure 194). In both cases we obtain the same expression for $(a')^2$:

$$(a')^2 = (b - c')^2 = (c' - b)^2 = b^2 - 2bc' + (c')^2.$$

Now the equality (*) can be rewritten as

$$a^2 = c^2 - (c')^2 + b^2 - 2bc' + (c')^2 = c^2 + b^2 - 2bc'.$$

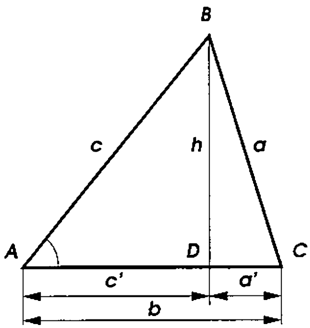

Figure 193

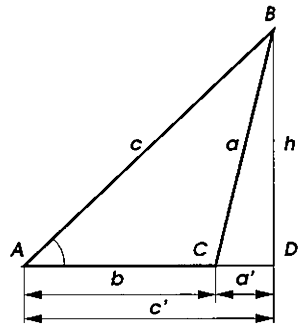

Figure 194

191. Theorem. ***In an obtuse triangle, the square of the side opposite to the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of (any) one of these two sides and the segment on the extension of this side between the vertex of the obtuse angle and the foot of the altitude drawn to this side.***

Let AB be the side of $\triangle ABC$ (Figure 194), opposite to the obtuse angle C, and BD the altitude dropped to the extension of another side, e.g. AC. It is required to prove that

$$AB^2 = AC^2 + BC^2 + 2AC \cdot CD,$$

or, using the abbreviated notation shown in Figure 194, that

$$c^2 = a^2 + b^2 + 2ba'.$$

From the right triangles ABD and CBD, we find:

$$c^2 = h^2 + (c')^2 = a^2 - (a')^2 + (a' + b)^2 =$$
$$a^2 - (a')^2 + (a')^2 + 2ba' + b^2 = \quad a^2 + b^2 + 2ba'.$$

192. Corollary. From the last three theorems, we conclude, that *the square of a side of a triangle is equal to, greater than, or smaller than the sum of the squares of the other two sides, depending on whether the angle opposite to this side is right, acute, or obtuse.*

Furthermore, this implies the converse statement: *an angle of a triangle turns out to be right, acute or obtuse, depending on whether the square of the opposite side is equal to, greater than, or smaller than the sum of the squares of the other two sides.*

193. Theorem. ***The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides*** (Figure 195).

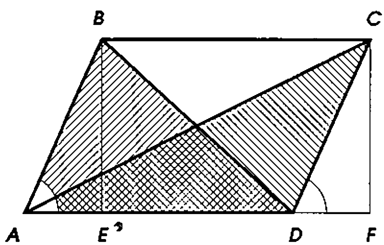

Figure 195

From the vertices B and C of a parallelogram $ABCD$, drop the perpendiculars BE and CF to the base AD. Then from the triangles ABD and ACD, we find:

$$BD^2 = AB^2 + AD^2 - 2AD \cdot AE, \quad AC^2 = AD^2 + CD^2 + 2AD \cdot DF.$$

The right triangles ABE and DCF are congruent, since they have congruent hypotenuses and congruent acute angles, and hence $AE = DF$. Having noticed this, add the two equalities found earlier. The summands $-2AD \cdot AE$ and $+2AD \cdot DF$ cancel out, and we get:

$$BD^2+AC^2 = AB^2+AD^2+AD^2+CD^2 = AB^2+BC^2+CD^2+AD^2.$$

194. We return to studying geometric means in a disk.

Theorem. ***If through a point*** (M, Figure 196), ***taken inside a disk, a chord*** (AB) ***and a diameter*** (CD) ***are drawn, then the product of the segments of the chord*** ($AM \cdot MB$) ***is equal to the product of the segments of the diameter*** ($CM \cdot MD$).

Drawing two auxiliary chords AC and BD, we obtain two triangles AMC and DMB (shaded in Figure 196) which are similar, since their angles A and D are congruent as inscribed intercepting the same arc BC, and the angles B and D are congruent as inscribed intercepting the same arc AD. From similarity of the triangles we derive: $AM : MD = CM : MB$, or equivalently

$$AM \cdot MB = CM \cdot MD.$$

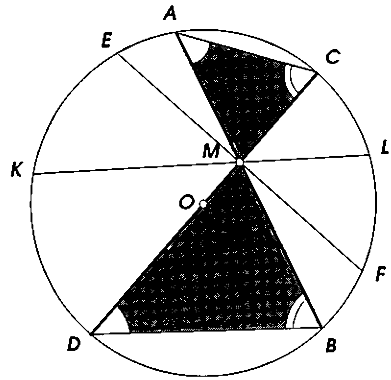

Figure 196

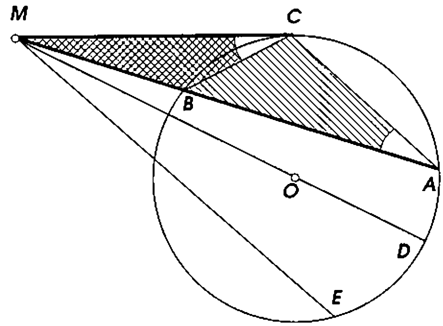

Figure 197

195. Corollaries. (1) *For all chords* (AB, EF, KL, Figure 196) *passing through the same point* (M) *inside a disk, the product of the segments of each chord is constant, i.e. it is the same for all such chords,* since for each chord it is equal to the product of the segments of the diameter.

(2) *The geometric mean between the segments* (AM and MB) *of a chord* (AB), *passing through a point* (M) *given inside a disk, is the segment* (EM or MF) *of the chord* (EF) *perpendicular to the diameter* (CD), *at the given point,* because the chord perpendicular to the diameter is bisected by it, and hence

$$EM = MF = \sqrt{AM \cdot MB}.$$

196. Theorem. ***The tangent*** (MC, Figure 197) ***from a point*** (M) ***taken outside a disk is the geometric mean between a secant*** (MA), ***drawn through the same point, and the exterior segment of the secant*** (MB).

Draw the auxiliary chords AC and BC, and consider two triangles MCA and MCB (shaded in Figure 197). They are similar because $\angle M$ is their common angle, and $\angle MCB = \angle BAC$ since each of them is measured by a half of the arc BC. Taking the sides MA and MC in ΔMCA, and the homologous sides MC and MB in ΔMCB, we obtain the proportion: $MA : MC = MC : MB$ and conclude, that the tangent MC is the geometric mean between the segments MA and MB of the secant.

197. Corollaries. (1) *The product of a secant* (MA, Figure 197), *passing through a point* (M) *outside a disk, and the exterior part of the secant* (MB) *is equal to the square of the tangent* (MC) *drawn from the same point,* i.e.:

$$MA \cdot MB = MC^2.$$

(2) *For all secants* (MA, MD, ME, Figure 197), *drawn from a point* (M) *given outside a disk, the product of each secant and the exterior segment of it, is constant, i.e. the product is the same for all such secants,* because for each secant this product is equal to the square MC^2 of the tangent drawn from the point M.

198. Theorem. ***The product of the diagonals of an inscribed quadrilateral is equal to the sum of the products of its opposite sides.***

This proposition is called **Ptolemy's theorem** after a Greek astronomer *Claudius Ptolemy* (85 – 165 A.D.) who discovered it.

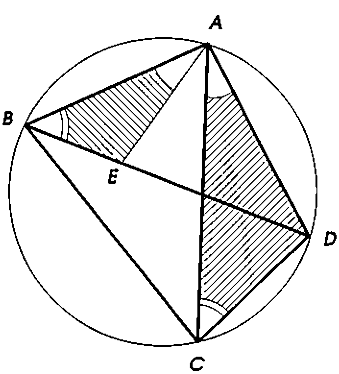

Figure 198

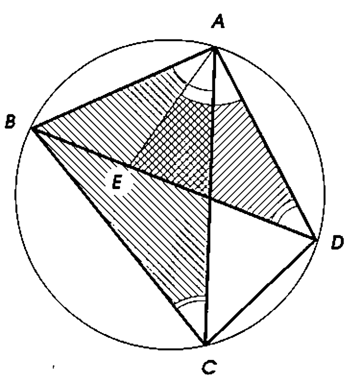

Figure 199

Let AC and BD be the diagonals of an inscribed quadrilateral $ABCD$ (Figure 198). It is required to prove that

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Construct the angle BAE congruent to $\angle DAC$, and let E be the intersection point of the side AE of this angle with the diagonal BD. The triangles ABE and ADC (shaded in Figure 198) are similar, since their angles B and C are congruent (as inscribed intercepting the same arc AD), and the angles at the common vertex A are congruent by construction. From the similarity, we find:

$$AB : AC = BE : CD, \quad \text{i.e.} \quad AC \cdot BE = AB \cdot CD.$$

Consider now another pair of triangles, namely $\triangle ABC$ and $\triangle AED$ (shaded in Figure 199). They are similar, since their angles BAC and DAE are congruent (as supplementing to $\angle BAD$ the angles congruent by construction), and the angles ACB and ADB are congruent as inscribed intercepting the same angle AB. We obtain:

$$BC : ED = AC : AD, \quad \text{i.e.} \quad AC \cdot ED = BC \cdot AD.$$

Summing the two equality, we find:

$$AC(BE + ED) = AB \cdot CD + BC \cdot AD, \text{ where } BE + ED = BD.$$

EXERCISES

Prove theorems:

400. If a diagonal divides a trapezoid into two similar triangles, then this diagonal is the geometric mean between the bases.

*401.** If two disks are tangent externally, then the segment of an external common tangent between the tangency points is the geometric mean between the diameters of the disks.

402. If a square is inscribed into a right triangle in such a way that one side of the square lies on the hypotenuse, then this side is the geometric mean between the two remaining segments of the hypotenuse.

*403.** If AB and CD are perpendicular chords in a circle of radius R, then $AC^2 + BD^2 = 4R^2$.

404. If two circles are concentric, then the sum of the squares of the distances from any point of one of them to the endpoints of any diameter of the other, is a fixed quantity.
Hint: See §193.

405. If two segments AB and CD (or the extensions of both segments) intersect at a point E, such that $AE \cdot EB = CE \cdot ED$, then the points A, B, C, D lie on the same circle.
Hint: This is the theorem converse to that of §195 (or §197).

*406.** In every $\triangle ABC$, the bisector AD satisfies $AD^2 = AB \cdot AC - DB \cdot DC$.
Hint: Extend the bisector to its intersection E with the circumscribed circle, and prove that $\triangle ABD$ is similar to $\triangle AEC$.

*407.** In every triangle, the ratio of the sum of the squares of all medians to the sum of the squares of all sides is equal to 5/4.

408. If an isosceles trapezoid has bases a and b, lateral sides c, and diagonals d, then $ab + c^2 = d^2$.

409. The diameter AB of a circle is extended past B, and at a point C on this extension $CD \perp AB$ is erected. If an arbitrary point M of this perpendicular is connected with A, and the other intersection point of AM with the circle is denoted A', then $AM \cdot AA'$ is a fixed quantity, i.e. it does not depend on the choice of M.

*410.** Given a circle $\mathcal{O}$ and two points A and B. Through these points, several circles are drawn such that each of them intersects with or is tangent to the circle $\mathcal{O}$. Prove that the chords connecting the intersection points of each of these circles, as well as the tangents at the points of tangency with the circle $\mathcal{O}$, intersect (when extended) at one point lying on the extension of AB.

411. Using the result of the previous problem, find a construction of the circle passing through two given points and tangent to a given circle.

Find the geometric locus of:

412. Points for which the sum of the squares of the distances to two given points is a fixed quantity.
Hint: See §193.

413. Points for which the difference of the squares of the distances from two given points is a fixed quantity.

Computation problems

414. Compute the legs of a right triangle if the altitude dropped from the vertex of the right angle divides the hypotenuse into two segments m and n.

415. Compute the legs of a right triangle if a point on the hypotenuse equidistant from the legs divides the hypotenuse into segments 15 and 20 cm long.

416. The centers of three pairwise tangent circles are vertices of a right triangle. Compute the smallest of the three radii if the other two are 6 and 4 cm.

417. From a point at a distance a from a circle, a tangent of length $2a$ is drawn. Compute the radius of the circle.

418. In the triangle ABC, the sides measure $AB = 7$, $BC = 15$, and $AC = 10$ units. Determine if the angle A is acute, right, or obtuse, and compute the altitude dropped from the vertex B.

419. Compute the radius of a circle which is tangent to two smaller sides of a triangle and whose center lies on the greatest side, if the sides are 10, 24 and 26 units long.

420. Through a point, which is 7 *cm* away from the center of a circle of radius 11 *cm*, a chord of length 18 *cm* is drawn. Compute the segments into which the point divides the chord.

421. From a point outside a disk, a tangent a and a secant are drawn. Compute the length of the secant if the ratio of its part outside the disk to the part inside the disk is equal to $m : n$.

422. Compute the base of an isosceles triangle with a lateral side 14 units and the median to this side 11 units.
Hint: Apply the theorem of §193.

*423.** Express medians of a triangle in terms of its sides.

*424.** Express altitudes of a triangle in terms of its sides.

*425.** Express bisectors of a triangle in terms of its sides.

*426.** A vertex of a triangle lies on the circle passing through the midpoints of the adjacent sides and the barycenter. Compute the median drawn from this vertex if the opposite side has length a.

*427.** In a triangle, the medians drawn to two sides of 6 and 8 *cm* long are perpendicular. Compute the third side.

7 Trigonometric functions

199. Trigonometric functions of acute angles. Let α be any acute angle (Figure 200). On one of its sides, take an arbitrary point M and drop the perpendicular MN from this point to the other side of the angle. Then we obtain a right triangle OMN. Take pairwise ratios of the sides of this triangle, namely:

$MN : OM$, i.e. the ratio of the leg opposite to the angle α, to the hypotenuse,

$ON : OM$, i.e. the ratio of the leg adjacent to the angle α, to the hypotenuse,

$MN : ON$, i.e. the ratio of the leg opposite to the angle α, to the leg adjacent to it, and the ratios reciprocal to them:

$$\frac{OM}{MN}, \quad \frac{OM}{ON}, \quad \frac{ON}{MN}.$$

The magnitude of each of these ratios depends neither on the position of the point M on the side of the angle, nor on the side of the angle the point M is taken on.

Indeed, if instead of the point M we take another point M' on the same side of the angle (or a point M'' on the other side of it), and drop the perpendiculars $M'N'$ (respectively $M''N''$) to the opposite side, then the right triangles thus formed: $\triangle OM'N'$ and $\triangle OM''N''$ will be similar to the triangle OMN, because α is their common acute angle. From the proportionality of homologous sides of similar triangles, we conclude:

$$\frac{MN}{ON} = \frac{M'N'}{ON'} = \frac{M''N''}{ON''}, \quad \frac{ON}{MN} = \frac{ON'}{M'N'} = \frac{ON''}{M''N''}, \quad \ldots$$

Therefore, the ratios in question do not change their values when the point M changes its position on one or the other side of the angle. Obviously, they do not change when the angle α is replaced by another angle congruent to it, but of course, they do change when the measure of the angle changes.

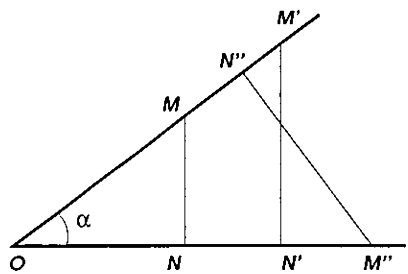

Figure 200

Thus, to acute angles of every given measure, there correspond quite definite values of each of these ratios, and we can therefore say that each of these ratios is a **function** of the angle only, and characterizes its magnitude.

All the above ratios are called **trigonometric functions** of the angle α. Out of the six ratios, the following four are used most often:

- the ratio of the leg opposite to the angle α, to the hypotenuse is called the **sine** of the angle α and is denoted $\mathbf{sin}\,\alpha$;

the ratio of the leg adjacent to the angle α, to the hypotenuse is called the **cosine** of the angle α and is denoted $\mathbf{cos}\,\alpha$;

the ratio of the leg opposite to the angle α, to the leg adjacent to it is called the **tangent** of the angle α and is denoted $\mathbf{tan}\,\alpha$;

the ratio of the adjacent leg to the opposite leg (i.e. the ratio reciprocal to $\tan\alpha$) is called the **cotangent** of the angle α and is denoted $\mathbf{cot}\,\alpha$.

Since each of the legs is smaller than the hypotenuse, the sine and cosine of any acute angle is a positive number smaller than 1, and since one of the legs can be greater, or smaller than the other leg, or equal to it, then the tangent and cotangent can be expressed by numbers greater than 1, smaller than 1, or equal to 1.

The remaining two ratios, namely the reciprocals of cosine and sine, are called respectively the **secant** and **cosecant** of the angle α, and are denoted respectively $\mathbf{sec}\,\alpha$ and $\mathbf{csc}\,\alpha$.

200. Constructing angles with given values of a trigonometric function.

(1) Suppose it is required *to construct an angle whose sine is equal to* 3/4. For this, one needs to construct a right triangle such that the ratio of one of its legs to the hypotenuse is equal to 3/4, and take the angle opposite to this leg. To construct such a triangle, take any small segment and mark the segment AB (Figure 201) congruent to 4 such segments. Then construct a semicircle on AB as a diameter, and draw an arc, of radius congruent to 3/4 of AB, centered at the point B. Let C be the intersection point of this arc with the semicircle. Connecting C with A and B we obtain a right triangle whose angle A will have the sine equal to 3/4.

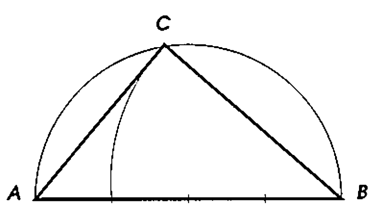

Figure 201

(2) *Construct an angle x satisfying the equation:* $\cos x = 0.7$. The problem is solved the same way as the previous one. Take the segment congruent to 10 arbitrary units for the hypotenuse AB (Figure 201), and congruent to 7 such units for AC. Then the angle A adjacent to this leg will be the required one.

(3) *Construct an angle x such that* $\tan x = 3/2$. For this, one needs to construct a right triangle such that one of its legs is $3/2$ times greater than the other. Draw a right angle (Figure 202), and mark a segment AB of arbitrary length on one of its sides, and the segment AC congruent to $\frac{3}{2}AB$ on the other. Connecting the points B and C, we obtain the angle B whose tangent is equal to $3/2$.

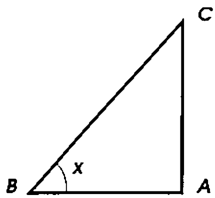

Figure 202

The same construction can be applied when the cotangent of the angle x is given, but the required angle in this case will be the one adjacent to the leg AC.

201. Behavior of trigonometric functions. It is convenient to describe the behavior of sine and cosine as the angle varies, assuming that the length of the hypotenuse remains fixed and equal to a unit of length, and only the legs vary. Taking the radius OA (Figure 203) equal to an arbitrary unit of length, describe a quarter-circle AM, and take any central angle $AOB = \alpha$. Dropping from B the perpendicular BC to the radius OA, we have:

$$\sin\alpha = \frac{BC}{OB} = \frac{BC}{1} = \text{ length of } BC,$$

$$\cos\alpha = \frac{OC}{OB} = \frac{OC}{1} = \text{ length of } OC.$$

Imagine now that the radius OB rotates about the center O in the direction pointed out by the arrow, starting from the position OA and finishing in the position OM. Then the angle α will increase from $0°$ to $90°$, passing through the values $\angle AOB$, $\angle AOB'$, $\angle AOB''$, etc. shown in Figure 203. In the process of rotation the length of the leg BC opposite to the angle α, will increase from 0 (for $\alpha = 0°$) to 1 (for $\alpha = 90°$), and the length of the leg OC adjacent to the angle α, will decrease from 1 (for $\alpha = 0°$) to 0 (for $\alpha = 90°$). Thus, ***when the angle α increases from $0°$ to $90°$, its sine increases from 0 to 1, and its cosine decreases from 1 to 0.***

Let us examine now the behavior of the tangent. Since the tangent is the ratio of the opposite leg to the adjacent leg, it is convenient to assume that the adjacent leg remains fixed and congruent to a unit of length, and the opposite leg varies with the angle. Take the segment OA congruent to a unit of length (Figure 204) for the fixed leg of the right triangle AOB, and start changing the acute angle $AOB = \alpha$. By definition,

$$\tan \alpha = \frac{AB}{OA} = \frac{AB}{1} = \text{ length of } AB.$$

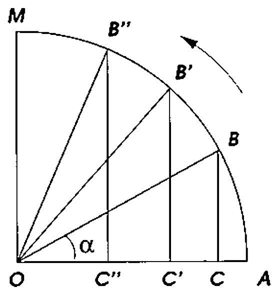

Figure 203

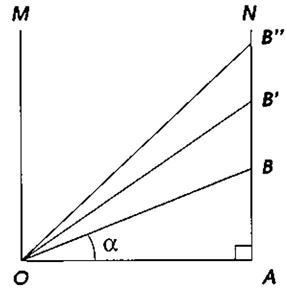

Figure 204

Imagine that the point B moves along the ray AN starting from the position A and going upward farther and farther, passing through the positions B', B'', etc. Then, as it is clear from Figure 204, both the angle α and its tangent will increase. When the point B coincides with A, the angle $\alpha = 0°$, and the tangent is also equal to 0. When the point B moves higher and higher, the angle α becomes closer and closer to $90°$, and the value of the tangent becomes greater and greater, exceeding any fixed number (i.e. grows indefinitely). In such cases one says that a function increases (or grows) to **infinity** (and expresses "infinity" by the symbol ∞). Thus, ***when the angle increases from*** $0°$ ***to*** $90°$***, its tangent increases from*** 0 ***to*** ∞.

From the definition of the cotangent as the quantity reciprocal to the tangent (i.e. $\cot x = 1/\tan x$), it follows that when the tangent increases from 0 to ∞, the cotangent decreases from ∞ to 0.

202. Trigonometric relationships in right triangles. We have defined trigonometric functions of acute angles as ratios of sides of right triangles associated with these angles. *Vice versa,* one can use the values of trigonometric functions in order to express metric relationships in right triangles.

(1) From a right triangle ABC (Figure 205), we find: $b/a = \sin B = \cos C$, $c/a = \cos B = \sin C$, and therefore

$$b = a\sin B = a\cos C, \quad c = a\cos B = a\sin C,$$

i.e. *a leg of a right triangle is equal to the product of the hypotenuse with the sine of the angle opposite to the leg, or with the cosine of the angle adjacent to it.*

(2) From the same triangle, we find: $b/c = \tan B = \cot C$ and $c/b = \cot B = \tan C$, and therefore

$$b = c\tan B = c\cot C, \quad c = b\cot B = b\tan C,$$

i.e. *a leg of a right triangle is equal to the product of the other leg with the tangent of the angle opposite to the former leg, or with the cotangent of the angle adjacent to it.*

Notice that $\angle B = 90° - \angle C$. It follows therefore that for any angle α

$$\cos\alpha = \sin(90° - \alpha), \ \sin\alpha = \cos(90° - \alpha),$$

$$\tan(90° - \alpha) = \cot\alpha, \ \cot(90° - \alpha) = \tan\alpha.$$

According to the Pythagorean theorem, we have $a^2 = b^2 + c^2$. Using this we arrive at the following fundamental identity relating the sine and cosine functions: ***the squares of the sine and cosine of the same angle add up to one:***

$$\sin^2\alpha + \cos^2\alpha = 1 \text{ for any angle } \alpha.$$

203. Some special values of trigonometric functions. Consider the right triangle ABC (Figure 206) such that its acute angle $B = 45°$. Then the other acute angle of this triangle is also equal to $45°$, i.e. the right triangle is isosceles: $b = c$. Therefore $a^2 = b^2 + c^2 = 2b^2$, and hence $b^2/a^2 = 1/2$, i.e. $b/a = 1/\sqrt{2}$. [6] Besides, $b/c = c/b = 1$. Thus

$$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}, \quad \tan 45° = \cot 45° = 1.$$

[6] According to §148, the hypotenuse a of an isosceles right triangle is incommensurable with its leg b. Since $a/b = \sqrt{2}$, we conclude that the number $\sqrt{2}$ is irrational.

Consider now the right triangle ABC (Figure 207) such that its acute angle $B = 30°$. According to the result of §81, the leg opposite to this angle is congruent to a half of the hypotenuse. Thus

$$\sin 30° = \cos 60° = \frac{1}{2}.$$

Now it follows from the Pythagorean theorem that

$$\cos 30° = \sin 60° = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Finally, since $\tan B = b : c = (1/2)a : (\sqrt{3}/2)a$, we have:

$$\tan 30° = \cot 60° = \frac{1}{2} : \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}, \quad \tan 60° = \cot 30° = \sqrt{3}.$$

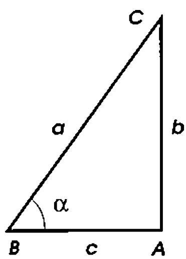

Figure 205

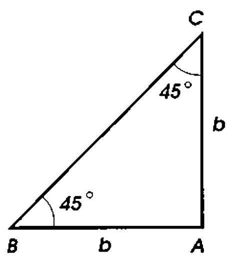

Figure 206

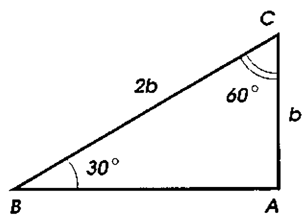

Figure 207

204. Trigonometric functions of obtuse angles. Definitions of trigonometric functions of acute angles can be successfully generalized to arbitrary angles using the concept of the number line and negative numbers, discussed in §153.

Consider an arbitrary central angle $BOA = \alpha$ (see Figure 208, where the angle α is shown obtuse) formed by a radius OB with the fixed radius OA. To define $\cos\alpha$, we first extend the radius OA to the infinite straight line, and identify the latter with the number line by taking the center O and the point A to represent the numbers 0 and 1 respectively. Then we drop the perpendicular from the endpoint of the radius B to the line OA. On the number line OA, the foot of this perpendicular represents a real number which is taken for the definition of the cosine of the angle α. To define $\sin\alpha$, we rotate the number line OA *counter-clockwise* through the angle of 90°, and thus obtain another number line, OP, perpendicular to OA. The foot of the perpendicular dropped from the point B to the line OP represents the number $\sin\alpha$. Translating the line OP we obtain a third number line AQ tangent to the circle at the point A. Then the intersection point of the extended line OB marks on the number line AQ the value of $\tan\alpha$. Finally, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$, are defined as the reciprocals of $\cos\alpha$, $\sin\alpha$, and $\tan\alpha$ respectively.

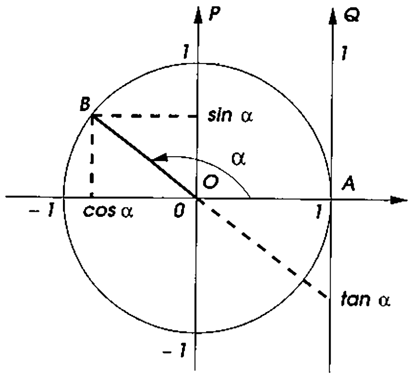

Figure 208

Some properties of trigonometric functions are obvious from Figure 208. For example, when the angle α is obtuse, the values $\cos\alpha$ and $\tan\alpha$ are negative, and $\sin\alpha$ positive. Moreover:

$$\sin\alpha = \sin(180^\circ - \alpha),\ \ \cos\alpha = -\cos(180^\circ - \alpha),$$

$$\tan\alpha = -\tan(180^\circ - \alpha),\ \ \cot\alpha = -\cot(180^\circ - \alpha).$$

205. The law of cosines. The notion of the cosine function for arbitrary angles allows one to unify the results of §190 and §191 and express the square of one side of a triangle in terms of the opposite angle and the other two sides, in a single formula known as the **law of cosines**.

Theorem. *The square of one side* (c, Figure 209) *of every triangle* (ABC) *is equal to the sum of the squares of the other two sides* (a and b) *minus twice the product of the latter two sides with the cosine of the angle* (C) *opposite to the former side:*

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Indeed, according to the result of §190 or §191, when the angle C is acute or obtuse, we have respectively:

$$c^2 = a^2 + b^2 - 2a \cdot CD, \quad \text{or} \quad c^2 = a^2 + b^2 + 2a \cdot CD, \qquad (*)$$

where CD is the distance from the vertex C to the perpendicular BD dropped from the vertex B to the opposite side. According to the definition of the number $\cos C$ (which is positive when $\angle C$ is acute, and negative when $\angle C$ is obtuse), $CD = b\cos C$ in the first case, and $CD = -b\cos C$ in the second. Substituting this value of CD into the corresponding equation $(*)$, we obtain the same resulting formula in both cases: $c^2 = a^2 + b^2 - 2ab\cos C$ as required. Finally, when the angle C is right, we have $\cos C = \cos 90° = 0$. Therefore the law of cosines turns in this case into the equality $c^2 = a^2 + b^2$, which holds true due to the Pythagorean theorem. Thus the law of cosines holds true for any triangle.

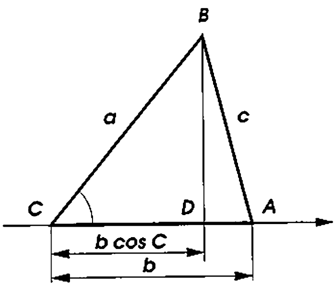

Figure 209

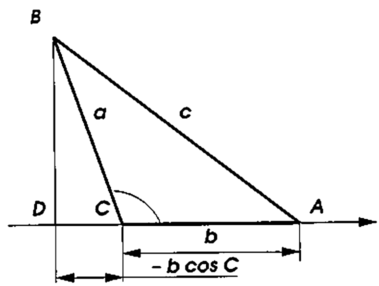

Figure 210

EXERCISES

428. Compute the values of the sine and cosine of the angles 90°, 120°, 135°, 150°, and 180°.

429. For which of the angles 0°, 90°, and 180° are the values of the functions **tan** and **cot** defined?

430. Compute the values of the tangent and cotangent of 120°, 135°, and 150°.

431. Prove that $\sin(\alpha + 90°) = \cos\alpha$, $\cos(\alpha + 90°) = \sin\alpha$.

432. Construct the angles α such that: (a) $\cos\alpha = 2/3$, (b) $\sin\alpha = -1/4$, (c) $\tan\alpha = 5/2$, (d) $\cot\alpha = -7$.

433. Compute two sides of a triangle, if the third side is a, and the angles adjacent to it are 45° and 15°.

434. Is the triangle with the sides 3, 7, and 8 *cm* acute, right, or obtuse? Compute the angle opposite to the middle side.

435. Compute the side AB of $\triangle ABC$ if $AC = 7$, $BC = 5$, and $\angle B = 120°$.

*436.** Compute the sine and cosine of: (a) 15°, (b) 22°30′.

*437.** Compute $\cos 18°$.

`Hint:` The bisector drawn to a lateral side of an isosceles triangle with the angle 36° at the vertex cuts off a triangle similar to the original one.

*438.** Prove that if from the endpoints of a diameter of a circle, two intersecting chords are drawn, then the sum of the products of each chord and the segment of it from the endpoint of the diameter to the intersection point is a constant quantity.

439. Prove that a side a of a triangle is expressed through the opposite angle and the radius R of the circumscribed circle as $a = 2R\sin A$.

440. Derive the **law of sines**: in every triangle, sides are proportional to the sines of the opposite angles.

*441.** Two right triangles lie on the opposite sides of their common hypotenuse h. Express the distance between the vertices of the right angles through h and the sines of acute angles of the triangles.

`Hint:` Apply Ptolemy's theorem.

442. Prove the **addition law** for the sine function:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

`Hint:` Apply the result of the previous problem.

*443.** On a given segment AB, a point M is chosen, and two congruent circles are drawn: through A and M, and M and B. Find the geometric locus of the second (i.e. other than M) intersection points of such circles.

8 Applications of algebra to geometry

206. The golden ratio. One says that a segment is divided in the **extreme and mean ratio** if the greater part is the geometric mean between the smaller part and the whole segment. In other words, the ratio of the whole segment to the greater part must be equal to the ratio of the greater part to the smaller one. [7] We will solve here the following construction problem:

[7] This ratio is known under many names, such as: the **golden ratio**, **golden section**, **golden mean**, and also the **divine proportion**.

Problem. *To divide a segment in the extreme and mean ratio.*

The problem will be solved if we find one of the two required parts, e.g. the greater one. Let us assume first that the problem in question is not about the construction of this part, but only about the *computation* of its length. Then the problem can be solved *algebraically*. Namely, if a denotes the length of the whole segment, and x the length of the greater required part, then the length of the other part is $a - x$, and the requirement of the problem is expressed by the equation:

$$x^2 = a(a - x), \text{ or } x^2 + ax - a^2 = 0.$$

Solving this quadratic equation we find two solutions:

$$x_1 = -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + a^2}, \; x_2 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 + a^2}.$$

We discard the second solution as negative, and simplify the first one:

$$x_1 = \sqrt{\frac{a^2}{4} + a^2} - \frac{a}{2} = \sqrt{\frac{5a^2}{4}} - \frac{a}{2} = \frac{\sqrt{5}a}{2} - \frac{a}{2} = \frac{\sqrt{5} - 1}{2} a.$$

Thus the problem has a unique solution. If we manage to *construct* a segment whose length is given by this formula, then our original problem will be solved. Thus the problem reduces to *constructing a given formula.*

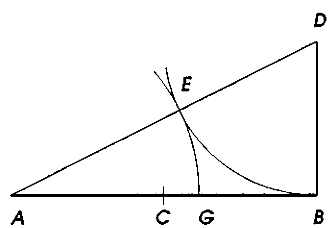

Figure 211

In fact it is more convenient to construct this formula in the form it had before the simplification. Considering the expression

$$\sqrt{\left(\frac{a}{2}\right)^2 + a^2},$$

we notice that it represents the length of the hypotenuse of a right triangle whose legs are $a/2$ and a. Constructing such a triangle and then subtracting $a/2$ from its hypotenuse, we find the segment x_1. Therefore the construction can be executed as follows.

Bisect the given segment $AB = a$ (Figure 211) at the point C. From the endpoint B, erect the perpendicular and mark on it the segment $BD = BC$. Connecting A and D we obtain a right triangle ABD whose legs are $AB = a$ and $BD = a/2$. Therefore its hypotenuse $AD = \sqrt{a^2 + (a/2)^2}$. To subtract $a/2$ from it, describe an arc BE of radius $BD = a/2$ centered at the point D. Then the remaining segment AE of the hypotenuse will be equal to x_1. Marking on AB the segment $AG = AE$, we obtain a point G, which divides the segment AB in the extreme and mean ratio.

207. The algebraic method of solving construction problems. We have solved the previous problem by way of *applying algebra to geometry*. This is a general method which can be described as follows. Firstly one determines which line segment is required in order to solve the problem, denotes known segments by $a, b, c, \ldots$, and the required segment by x, and expresses relationships between these quantities in the form of an algebraic equation, using requirements of the problem and known theorems. Next, applying the methods of algebra, one solves the equation, and then studies the solution formula thus found, i.e. determines for which data the solution exists, and how many solutions there are. Finally, one constructs the solution formula, i.e. describes a construction by straightedge and compass of a segment whose length is expressed by this formula.

Thus the algebraic method of solving geometric construction problems, generally speaking, consists of four steps: (i) deriving an equation, (ii) solving it, (iii) studying the solution formula, (iv) constructing it.

Sometimes a problem reduces to finding several line segments. Then one denotes their lengths by several letters $x, y, z, \ldots$, and seeks a system of as many equations as there are unknowns.

208. Construction of elementary formulas. Suppose that solving a construction problem by the algebraic method we arrive at a solution formula which expresses a required length x through given lengths $a, b, c, \ldots$ using only the arithmetic operations of addition, subtraction, multiplication and division, and the operation of extracting square roots. We will show here, how to construct such a formula by straightedge and compass.

First, one of the given segments, e.g. a, can be taken for the unit of length. We may assume therefore that all segments are represented by numbers. Respectively, the task of constructing the formulas expressing the required segment through given segments is reduced to the problem of constructing the required number x expressed through the given numbers $a = 1, b, c, \ldots$ by means of the four arithmetic operations and by extracting square roots. Thus it suffices to show how to obtain the result of these five elementary operations with given numbers, using straightedge and compass.

(1) Addition and subtraction of numbers represented by given segments can be easily done by marking the segments on the number line (using compass).

(2) Multiplication and division can be done on the basis of Thales' theorem by intersecting sides of an angle by parallel lines, as shown in Figure 212. Namely, the proportions

$$\frac{x}{b} = \frac{c}{1}, \text{ and } \frac{x}{1} = \frac{b}{c}$$

are equivalent to $x = bc$ and $x = b/c$ respectively.

(3) To extract the square root x of a given number b, it suffices to construct the geometric mean between b and 1 as shown in Figure 213.

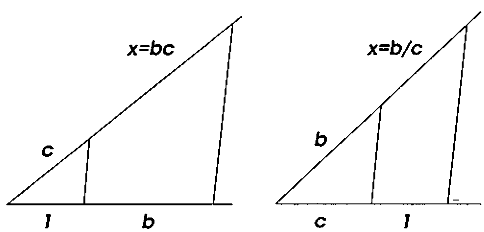

Figure 212

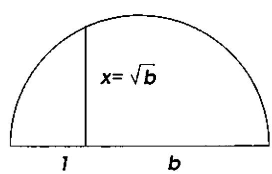

Figure 213

Thus, ***any algebraic expressions involving only arithmetic operations with and square roots of given numbers can be constructed by straightedge and compass.***

Remark. Conversely, as we will see in §213, using straightedge and compass one can construct only those algebraic expressions which can be obtained from given numbers by a finite succession of arithmetic operations and extraction of square roots.

EXERCISES

444. Construct the angle $\frac{1}{10}d$.

445. Construct an isosceles triangle such that the bisector of an angle adjacent to the base cuts off a triangle similar to it.

446. Given three segments a, b, and c, construct a fourth segment x such that $x : c = a^2 : b^2$.

447. Construct segments expressed by the formulas: (a) $x = abc/de$, (b) $x = \sqrt{a^2 + bc}$

448. Given the base a and the altitude h of an acute triangle, compute the side x of the square inscribed into the triangle, i.e. such that one side of the square lies on the base, and the opposite vertices on the lateral sides of the triangle.

449. A common tangent is drawn to two disks which have the distance d between the centers, and the radii R and r. Compute the position of the intersection point of this tangent with the line of centers, when the point lies: (i) to one side of both centers, or (ii) between them.

450. Prove that if two medians in a triangle are congruent, then the triangle is isosceles.
`Hint`: Use the algebraic method and §193.

451. In the exterior of a given disk, find a point such that the tangent from this point to the disk is equal to a half of the secant drawn from this point through the center.
`Hint`: Apply the algebraic method.

452. Through a given point outside a given disk, construct a secant that is divided by the circle in a given ratio.

453. Inscribe a circle into a given sector.

*454.** Construct a triangle given its altitudes.
`Hint`: First derive from similarity of triangles that the altitudes h_a, h_b, h_c are *inversely proportional* to the respective sides a, b, c, i.e. that $h_a : h_b : h_c = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$.

9 Coordinates

209. Cartesian coordinates. We saw in §153 how to identify points of a straight line with real numbers. It turns out that points of a plane can similarly be identified with *ordered pairs* of real numbers. One important way of doing this is to introduce **Cartesian coordinates.**[8] To construct a Cartesian **coordinate system** on

[8]The term *Cartesian* originates from *Cartesius*, the Latinized name of René Descartes (1696 – 1650), the French philosopher who introduced into geometry the systematic use of algebra.

the plane, pick a point O (Figure 214) and two perpendicular lines passing through it. Then pick a unit of length, and mark segments OA and OB of unit length on the first and second line respectively. The point O is called the **origin** of the coordinate system, and the infinite straight lines OA and OB the 1st and the 2nd **coordinate axes** respectively.

Next, identify each of the coordinate axes with the number line by choosing the origin to represent the number 0 on each of them, and the point A (respectively B) to represent the number 1 on the 1st (respectively the 2nd) axis.

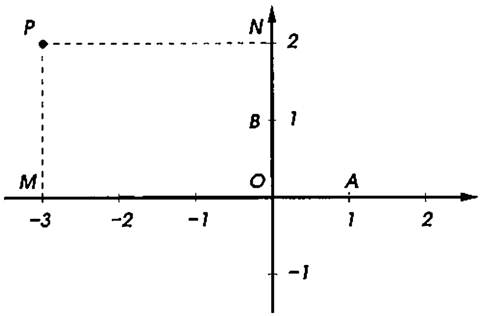

Figure 214

Now, given a coordinate system, to any point P on the plane, we associate an ordered pair (x, y) of real numbers called respectively the 1st and the 2nd **coordinate** of P. Namely, we draw through P two lines PN and PM, parallel to the coordinate axes OA and OB respectively. The intersection point M (respectively N) of the line OM (respectively ON) with the 1st (respectively the 2nd) coordinate axis represents on this axis a real number, which we take for x (respectively y). For instance, the point P in Figure 214 has the coordinates $x = -3$, and $y = 2$. *Vice versa,* the point P can be recovered from its coordinates (x, y) unambiguously. Namely, mark on the 1st and 2nd coordinate axes the points representing the numbers x and y respectively, and erect perpendiculars to the axes from these points. Obviously, P is the intersection point of these perpendiculars. Therefore we have established a correspondence between points of the plane and ordered pairs of their coordinates. Clearly, the coordinates in this construction can be arbitrary real numbers, and we will write $P(x, y)$ for a point P whose 1st and 2nd coordinates are given by the numbers x and y respectively (e.g. $P(-3, 2)$ is the point denoted P which has the coordinates $x = -3$ and $y = 2$).

210. The coordinate distance formula.

Problem. *To compute the length of the segment between two points $P(x,y)$ and $P'(x',y')$ with given Cartesian coordinates* (Figure 215).

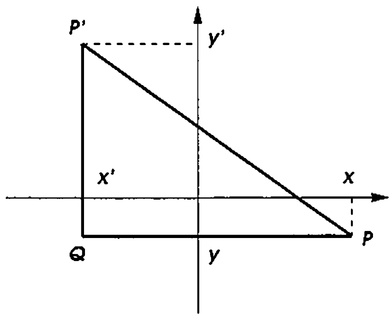

Figure 215

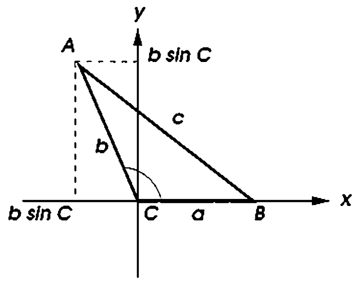

Figure 216

The lines PQ and $P'Q$, parallel to the 1st and 2nd coordinate axes respectively, are perpendicular (since the coordinate axes are), and therefore intersect at some point Q. Suppose that the segment PP' is not parallel to either of the coordinate axes. Then PP' is the hypotenuse of the right triangle PQP'. Applying the Pythagorean theorem, we find the distance between $P(x,y)$ and $P'(x',y')$:

$$PP' = \sqrt{(x-x')^2 + (y-y')^2}.$$

In the special case when the segment PP' is parallel to one of the coordinate axes, the right triangle PQP' degenerates into this segment, but it is easy to check that the above distance formula remains true (because in this case either $x = x'$, or $y = y'$).

211. The method of coordinates. One can successfully use coordinates to solve geometric problems. Here is an example.

Problem. *To re-prove the law of cosines using coordinates.*

In $\triangle ABC$, let a, b, and c be the sides opposite to the vertices A, B, and C respectively. It is required to prove that

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Pick a Cartesian coordinate system in such a way that the origin is the vertex C (Figure 216), the positive ray of the 1st coordinate axis contains the side CB, and the positive ray of the 2nd coordinate axis lies on the same side of the line CB as the vertex A. Then the vertices C, B, and A have coordinates respectively: $(0,0)$, $(a,0)$ (by construction), and $(b\cos C, b\sin C)$ (by the definition of sine and cosine). The distance c between the vertices A and B can be computed using the coordinate distance formula of §210 with $(x,y) = (b\cos C, b\sin C)$ and $(x', y') = (a, 0)$, i.e.

$$c^2 = (b\cos C - a)^2 + (b\sin C)^2 = b^2\cos^2 C - 2ab\cos C + a^2 + b^2\sin^2 C.$$

The first and the last summands here add up to b^2, since $\cos^2 C + \sin^2 C = 1$. We obtain therefore $c^2 = a^2 + b^2 - 2ab\cos C$ as required.

212. Geometric loci and their equations. The geometric locus of all points, whose coordinates (x,y) satisfy a certain equation, is said to be *described by this equation*, and is called the **solution locus** of it. Many familiar geometric loci can be described in coordinates as solution loci of suitable equations. We discuss here the equations of straight lines and circles.

Problem. *To find the geometric locus of points $P(x,y)$ whose coordinates satisfy the equation $\alpha x + \beta y = \gamma$, where α, β, and γ are given numbers.*

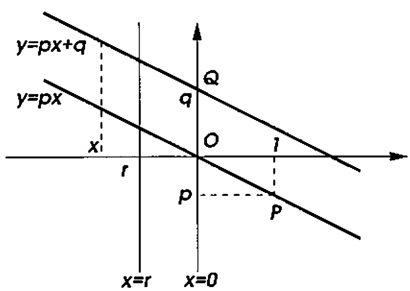

Figure 217

When $\alpha = \beta = 0$, the left hand side of the equation is equal to 0, and therefore the geometric locus in question contains all points of the plane when $\gamma = 0$, and contains no points when $\gamma \neq 0$. So, let us assume that at least one of the coefficients α, β is non-zero. In this case we claim that *the points whose coordinates (x,y) satisfy the equation $\alpha x + \beta y = \gamma$ form a straight line.* To see this, we divide the equation by β, assuming that $\beta \neq 0$, and obtain a new equation $y = px + q$, where $p = -\alpha/\beta$, and $q = \gamma/\beta$. Of course, multiplication or division of an equation by a non-zero number does not change the locus of points whose coordinates satisfy the equation. Thus we need to show that the locus of solutions of the new equation is a straight line.

Consider first the case when $q = 0$. Points satisfying the equation $y = px$ are exactly the points with coordinates (x, y) of the form (x, px). The locus of such points contains exactly one point for each value of x and includes: the origin O (Figure 217) whose coordinates are $(x, y) = (0, 0)$; the point P with coordinates $(x, y) = (1, p)$; all points homothetic to P with respect to the center O and with arbitrary homothety coefficients x (positive or negative). Thus the locus is a straight line passing through the origin (and non-parallel to the 2nd coordinate axis).

When $q \neq 0$, we note that the locus does not contain the origin, but instead contains the point Q with coordinates $(x, y) = (0, q)$. Moreover, each point (x, px) of the line $y = px$ is replaced by the new point $(x, px + q)$, obtained from the old one by translation in the direction of the segment OQ. Thus the solutions to the equation $y = px + q$ form the line parallel to the line $y = px$ and passing through the point $Q(0, q)$.

Finally, when $\beta = 0$, but $\alpha \neq 0$, we can divide the equation by α and obtain a new equation $x = r$, where $r = \gamma/\alpha$. When $r = 0$, the solutions locus is the 2nd coordinate axis, and when $r \neq 0$, the solutions $(x, y) = (r, y)$ form a straight line parallel to the 2nd coordinate axis and passing through the point $(r, 0)$.

Since any line on the plane is parallel to one of the lines passing through the origin, we conclude that, *vice versa, any straight line on the plane is the solution locus to an equation of the form $\alpha x + \beta y = \gamma$, where at least one of the coefficients α, β is non-zero.*

Problem. *To find an equation of the straight line passing through two points $P'(x', y')$ and $P''(x'', y'')$ with given coordinates.*

Let $P(x, y)$ (Figure 218) be a third point on the line passing through P' and P''. Then P is homothetic to P'' with respect to the center P' (and with an arbitrary homothety coefficient which can be positive or negative). The corresponding homothety of right triangles (shaded on Figure 218) yields the following proportion:

$$\frac{x - x'}{x'' - x'} = \frac{y - y'}{y'' - y'}.$$

This equation makes sense whenever $x' \neq x''$ and $y' \neq y''$ (i.e. when the segment $P'P''$ is not parallel to any coordinate axis), and can be rewritten in the form $\alpha x + \beta y = \gamma$ with

$$\alpha = \frac{1}{x'' - x'}, \quad \beta = -\frac{1}{y'' - y'}, \quad \text{and } \gamma = \frac{x'}{x'' = x'} - \frac{y'}{y'' - y'}.$$

When $x' = x''$ (or $y' = y''$), the line is parallel to the 2nd (respectively the 1st) coordinate axis, and has an equation $x = x'$ (respectively $y = y'$).

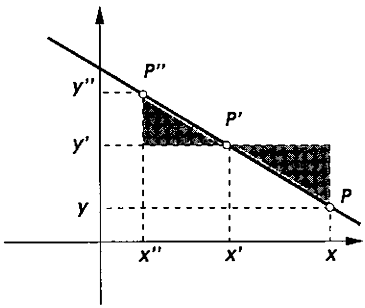

Figure 218

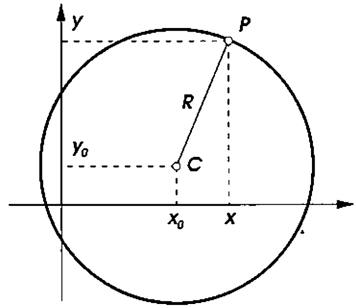

Figure 219

Problem. *To find an equation of the circle of a given radius R and centered at a given point $C(x_0, y_0)$* (Figure 219).

The circle consists of all points $P(x, y)$ whose distance to C is equal to R. Using the coordinate distance formula, we obtain the equation $\sqrt{(x - x_0)^2 + (y - y_0)^2} = R$ or, equivalently,

$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$

213. Constructibility. We saw in §206 that geometric quantities expressible in terms of given ones by means of elementary formulas, i.e. by arithmetic operations and extraction of square roots, can be constructed by straightedge and compass. Now we can show, using the method of coordinates, that the converse proposition holds true:

Every geometric quantity which can be constructed from given ones by means of straightedge and compass, can be expressed in terms of the given quantities using only arithmetic operations and extraction of square roots.

The starting point is the observation that a construction by straightedge and compass is a finite succession of the following **elementary constructions**:

(i) drawing a new line through two given points;

(ii) drawing a new circle, given its center and the radius;

(iii) drawing a circle, given one of its points and the center;

(iv) constructing a new point by intersecting two given non-parallel lines;

(v) constructing a new point by intersecting a given line with a given circle;

(vi) constructing a new point by intersecting two given non-concentric circles.

We can equip the plane with a Cartesian coordinate system and assume that "given points" are points whose coordinates are given real numbers, and "given radii" are segments whose lengths are given. Thus it suffices to show that *the elementary constructions (i) — (vi) give rise to points which have coordinates expressible through given numbers by elementary formulas, or to lines and circles whose equations have coefficients expressible by elementary formulas.*

(i) As we have seen in §212, the line passing through two given points has an equation whose coefficients are expressed through the coordinates of these points by means of arithmetic operations.

(ii) Similarly, the circle with given center and radius has an equation whose coefficients are arithmetic expressions of the coordinates of the center and the radius.

(iii) According to §210, the distance between two given points is expressed through their coordinates as the square root of an expression involving only arithmetic operations. Thus the required conclusion follows from (ii).

(iv) To find the coordinates of the intersection point of two non-parallel lines, whose equations have given coefficients (e.g. the lines with the equations $2x - 3y = 1$ and $6x + 5y = 7$), we can use one of the equations to express one of the coordinates through the other one (e.g. express $x = (1+3y)/2 = 0.5+1.5y$ from the first equation), substitute the expression into the other equation (i.e. write $6(0.5 + 1.5y)+5y = 7$, or $8y = 4$), find the value of the other coordinate from the resulting equation ($y = 4/8 = 0.5$), and then compute the value of the former coordinate ($x = 0.5+1.5 \times 0.5 = 1.25$). This procedure involves only arithmetic operations with the given coefficients.

(v) To find intersection points of a line and a circle with the given equations

$$\alpha x + \beta y = \gamma, \text{ and } (x - x_0)^2 + (y - y_0)^2 = R^2,$$

we can express one of the coordinates through the other from the first equation (say, $y = px+q$, if $\beta \neq 0$), and substitute the result into the second equation. The resulting equation $(x-x_0)^2 + (px+q-y_0)^2 = R^2$ is easily transformed (by squaring explicitly the expression in parentheses and reducing similar terms) to the form

$$Ax^2 + Bx + C = 0,$$

where A, B, and C are arithmetic expressions of the given numbers α, β, γ, x_0, y_0, and R. As it is well-known from algebra, solutions of this equation are expressed through the coefficients A, B, and C, using only arithmetic operations and square roots, namely (if $A \neq 0$):

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Thus the coordinate x of an intersection point, and therefore the other coordinate $y = px + q$ as well, are obtained from the given numbers using only successions of elementary formulas.

(vi) Consider equations of two circles with given centers and radii:

$$(x - x_1)^2 + (y - y_1)^2 = R_1^2 \quad \text{and} \quad (x - x_2)^2 + (y - y_2)^2 = R_2^2.$$

The coordinates (x, y) of intersection points of the circles must satisfy both equations. Squaring explicitly the parenthesis we rewrite the equations in this way:

$$x^2 + y^2 - 2x_1x - 2y_1y = R_1^2 - x_1^2 - y_1^2,$$

$$x^2 + y^2 - 2x_2x - 2y_2y = R_2^2 - x_2^2 - y_2^2.$$

We can replace the second equation in this system by the difference of the second and the first equation.The result has the form

$$2(x_1 - x_2)x + 2(y_1 - y_2)y = \gamma, \qquad (*)$$

where γ is an arithmetic expression of given numbers. Since the two circles are non-concentric, the differences $x_1 - x_2$ and $y_1 - y_2$) cannot both be zero, and hence the equation $(*)$ describes a straight line. The problem (vi) of intersecting two non-concentric circles with given centers and radii is reduced therefore to the problem (v) of intersecting a line and a circle whose equations have given coefficients. Thus coordinates of intersection points of two given non-concentric circles are also obtained by successions of elementary operations with given numbers.

Remark. As we know, two circles can have at most two common points (§104), and such points must lie on a line perpendicular to the line of centers (§117). Our result shows how to express an equation of this line (namely (*)) in terms of the radii and the centers of the circles.

EXERCISES

455. Prove that the triangle with the vertices $A(2,-3)$, $B(6,4)$, and $C(10,-4)$ is isosceles. Is it acute, right or obtuse?

456. Prove that the triangle with the vertices $A(-3,1)$, $B(4,2)$, and $C(3,-1)$ is right.

457. Find coordinates of the midpoint of a segment in terms of coordinates of its endpoints.

458. Prove that each coordinate of the barycenter of a triangle is the arithmetic average of the corresponding coordinates of the vertices.

459. The diagonals of a square $ABCD$ intersect at the origin. Find coordinates of B, C, and D, if the coordinates of A are given.

460. Prove that the sum of the squares of distances from the vertices of a given square to a line passing through its center is constant.

461. Compute the distance between the incenter and barycenter of a right triangle with legs 9 and 12 *cm*.

462. Prove that for any rectangle $ABCD$ and any point P, we have $PA^2 + PC^2 = PB^2 + PD^2$.

463. Can a triangle be equilateral, if distances from its vertices to two given perpendicular lines are expressed by whole numbers?

464. Using the method of coordinates, re-prove the result of §193: the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals.

465. Prove that the geometric locus of points $P(x,y)$ described by the equation $x^2 + y^2 = 6x + 8y$ is a circle, and find its center and radius.

466. Using the method of coordinates, re-prove Apollonius' theorem that the geometric locus of points from which the distances to two given points have a given ratio $m : n$, not equal to 1, is a circle.

*467.** Prove that if three pairwise intersecting circles are given, then the three lines, each passing through the intersection points of two of the circles, are concurrent.

Chapter 4

REGULAR POLYGONS AND CIRCUMFERENCE

1 Regular polygons

214. Definitions. A polygon (§31) is called **regular** if all of its sides are congruent and all of its interior angles are congruent. More generally, a broken line (not necessarily closed) is called **regular**, if all of its sides are congruent, and all of its angles on the same side of the broken line are congruent. For example, the broken line in Figure 220 has congruent sides and angles, but it is not regular since some of the congruent angles are situated on the opposite sides of the line. The five-point star in Figure 221 is an example of a

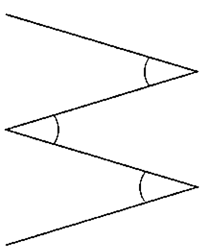

Figure 220

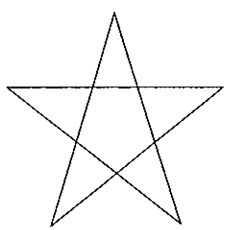

Figure 221

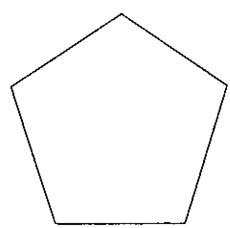

Figure 222

closed regular broken line, since all of its 5 sides are congruent as all of its 5 interior angles are. But we do not consider it a polygon, because it has self-intersections. An example of a regular polygon is the pentagon shown in Figure 222.

Forthcoming theorems show that construction of regular polygons is closely related to division of circles into congruent parts.

215. Theorem. ***If a circle is divided into a certain number (greater than 2) of congruent parts, then:***

(1) ***connecting every two consecutive division points by chords, we obtain a regular polygon, inscribed into the circle;***

(2) ***drawing tangents to the circle at all the division points and extending each of them up to the intersection points with the tangents at the nearest division points, we obtain a regular polygon circumscribed about the circle.***

Let the circle (Figure 223) be divided at the points A, B, C, etc. into several congruent parts, and through these points the chords AB, BC, etc. are drawn, and the tangents MBN, NCP, etc. Then the inscribed polygon $ABCDEF$ is regular, because all its sides are congruent (as chords subtending congruent arcs), and all of its angles are congruent (as inscribed angles, intercepting congruent arcs).

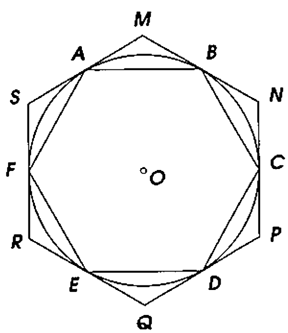

Figure 223

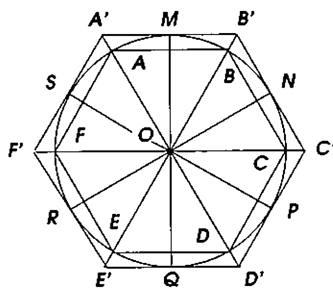

Figure 224

In order to prove regularity of the circumscribed polygon $MNPQRS$, consider the triangles AMB, BNC, etc. The bases AB, BC, etc. of these triangles are congruent, and the angles adjacent to the bases are also congruent because each of them has the same measure (since an angle formed by a tangent and a chord measures a half of the arc contained inside the angle). Thus all these triangles are isosceles and congruent to each other, and hence $MN = NP = \dots$, and $\angle M = \angle N = \dots$, i.e. the polygon $MNPQRS$ is regular.

216. Remark. If from the center O (Figure 224), we drop to the chords AB, BC, etc. perpendiculars and extend them up to the intersections with the circle at the points M, N, etc., then these points bisect all the arcs and chords, and therefore divide the circle into congruent parts. Therefore, if through the points M, N, etc. we draw tangents to the circle up to their mutual intersection as explained earlier, then we obtain another circumscribed regular polygon $A'B'C'D'E'F'$, whose sides are parallel to the sides of the inscribed one. Each pair of vertices: A and A', B and B', etc., lie on the same ray with the center, namely on the bisector of the angle MON and other such angles.

217. Theorem. ***If a polygon is regular, then;***

(1) ***it is possible to circumscribe it by a circle;***

(2) ***it is possible to inscribe a circle into it.***

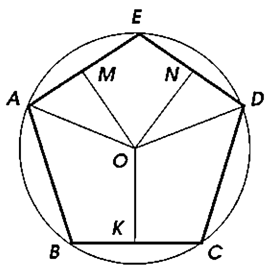

Figure 225

(1) Draw a circle through any three consecutive vertices A, B, and C (Figure 225) of a regular polygon $ABCDE$ and prove that it will pass through the next vertex D. For this, drop from the center O the perpendicular OK to the chord BC and connect O with A and D. Rotate the quadrilateral $ABKO$ in space about the side OK so that it falls onto the quadrilateral $DCKO$. Then the line KB will fall onto KC (due to equality of the right angles at the point K), and B will merge with C (since the chord BC is bisected at K). Then the side BA will fall onto CD (due to equality of the angles B and C), and finally, the point A will merge with D (since $BA = CD$). This implies that OA will merge with OD, and therefore the points A and D are equidistant from the center. Thus the point D lies on the circle passing through A, B, and C. Similarly, this circle, which passes through B, C, and D, will pass through the next vertex E, etc; hence it passes through all vertices of the polygon.

(2) It follows from part (1) that sides of a regular polygon can be considered as congruent chords of the same circle. But such chords are equidistant from the center, and therefore the perpendiculars OM, ON, etc., dropped from O to the sides of the polygon, are congruent to each other. Thus the circle described by the radius OM from the center O is inscribed into the polygon $ABCDE$.

218. Corollaries. (1) *Any regular polygon* ($ABCDE$, Figure 226) *is convex,* i.e. it lies on one side of each line extending any of its sides.

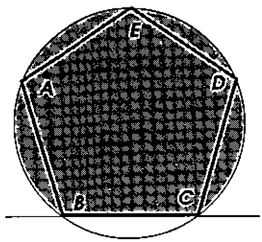

Figure 226

Indeed, extend, for instance, the side BC and note that it divides the circumscribed circle into two arcs. Since all vertices of the polygon lie on this circle, they must all lie on one of these arcs (because otherwise the broken line $BAEDC$ would intersect the segment BC, in contradiction to our definition of a polygon). Thus the whole regular polygon lies in the disk segment ($BAEDC$ in Figure 226) enclosed between this arc and the line BC, and hence on one side of this line.

(2) As it is clear from the proof of the theorem, *the inscribed and circumscribed circles of a regular polygon are concentric.*

219. Definitions. The common center of the inscribed and circumscribed circle of a regular polygon is called the **center** of this polygon. It lies on each angle bisector of the polygon and on each perpendicular bisector to its sides. Therefore, in order to locate the center of a regular polygon, it suffices to intersect two of its angle bisectors, or two perpendicular bisectors of its sides, or one of those angle bisectors with one of those perpendiculars.

The radius of the circle circumscribed about a regular polygon is called the **radius** of the polygon, and the radius of the inscribed circle its **apothem**. The angle between two radii drawn to the endpoints of any side is called a **central angle** of the regular polygon. There are as many such angles as there are sides, and they all are congruent (as central angles corresponding to congruent arcs).

Since the sum of all the central angles is $4d$ (or $360°$), then each of them is $4d/n$ (or $360°/n$), where n denotes the number of sides of the regular polygon. Thus, the central angle of a regular hexagon is $360°/6 = 60°$, of a regular **octagon** (i.e. 8-gon) $360°/8 = 45°$, etc.

220. Theorem. ***Regular polygons with the same number of sides are similar, and their sides have the same ratio as their radii or apothems.***

To prove the similarity of regular n-gons $ABCDEF$ and $A'B'C'D'E'F'$ (Figure 227), it suffices to show that their angles are congruent and their sides are proportional. The angles are congruent because they have the same measure, namely $2d(n-2)/n$ (see §82). Since $AB = BC = CD = \dots$ and $A'B' = B'C' = C'D' = \dots$, it is obvious that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \dots,$$

i.e. that the sides of such polygons are proportional.

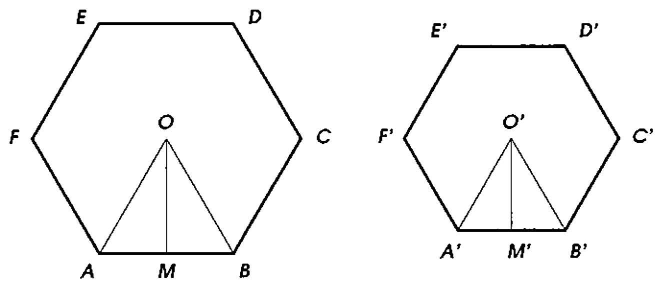

Figure 227

Let O and O' (Figure 227) be the centers of the given regular polygons, OA and $O'A'$ be their radii, and OM and OM' be their apothems. The triangles OAB and $O'A'B'$ are similar, since the angles of one of them are respectively congruent to the angles of the other. It follows from the similarity that

$$\frac{AB}{A'B'} = \frac{OA}{O'A'} = \frac{OM}{O'M'}.$$

Corollary. Since the perimeters of similar polygons have the same ratio as their homologous sides (§169), then *perimeters of regular n-gons have the same ratio as their radii or apothems.*

Example. Let a and b be the sides of regular polygons with the same number of sides, respectively inscribed into and circumscribed about the same circle of radius R. Then the apothem of the circumscribed polygon is R. From the right triangle AOM (Figure 227), we find the apothem OM of the inscribed polygon: $OM^2 = R^2 - (a/2)^2 = R^2 - a^2/4$. Since the inscribed and circumscribed polygons are similar, we can write the proportion between their sides and apothems:

$$\frac{b}{a} = \frac{R}{\sqrt{R^2 - a^2/4}}, \text{ i.e. } b = \frac{aR}{\sqrt{R^2 - a^2/4}}.$$

Thus we obtain a formula expressing the side of the circumscribed regular polygon through the side and the radius of the corresponding inscribed regular polygon.

221. Symmetries of regular polygons. In the circumscribed circle of a regular polygon, draw through any vertex C the diameter CN (Figure 228). It divides the circle and the polygon into two parts. Imagine that one of these parts (say, the left one) is rotated in space about the diameter so that it falls onto the other (i.e. right) part. Then one semicircle will merge with the other semicircle, the arc CB with the arc CD (due to the congruence of these arcs), the arc BA with the arc DE (for the same reason), etc., and therefore the chord BC will merge with the chord CD, the chord AB with the chord DE, etc. Thus *the diameter of the circumscribed circle drawn through any vertex of a regular polygon is an axis of symmetry of this polygon.* As a consequence of this, each pair of the vertices such as B and D, A and E, etc., lie on the same perpendicular to the diameter CN and at the same distance from it.

Draw also the diameter MN (Figure 229) of the circumscribed circle, which is perpendicular to any side CD of the regular polygon. This diameter also divides the circle and the polygon into two parts. Rotating one of them in space about the diameter until it falls onto the other part, we find out that one part of the polygon will merge with the other part. We conclude that *a diameter of the circumscribed circle perpendicular to any side of a regular polygon is an axis of symmetry of this polygon.*

Consequently, each pair of vertices such as B and E, A and F, etc., lie on the same perpendicular to the diameter MN and at the same distance from it.

If the number of sides of the regular polygon is *even*, then the diameter drawn through any vertex of the polygon also passes through the opposite vertex, and the diameter perpendicular to any side of the polygon is also perpendicular to the opposite side of it. If the number of sides is *odd*, then the diameter passing through any vertex is perpendicular to the opposite side, and conversely, the diameter perpendicular to any side of such a regular polygon passes through the opposite vertex. For example, the regular hexagon has 6 axes of symmetry: 3 axes passing through the vertices, and 3 axes perpendicular to the sides; the regular pentagon has 5 symmetry axes, each one passing through a vertex and perpendicular to the opposite side.

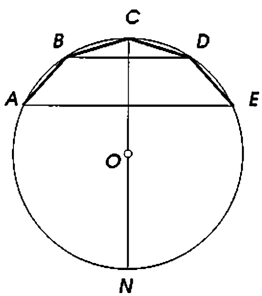

Figure 228

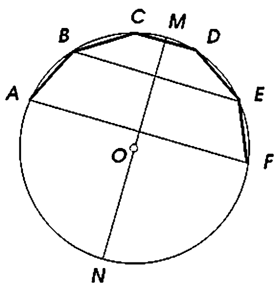

Figure 229

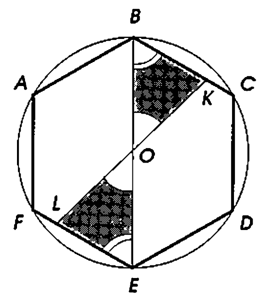

Figure 230

Any regular polygon with an *even* number of sides also has a *center of symmetry* which coincides with the center of the polygon (Figure 230). Indeed, any straight line KL, connecting two points on the boundary of the polygon and passing through its center O is bisected by it (as it is seen from the congruence of the triangles OBK and OEL shaded in Figure 230).

Finally, we can identify a regular n-gon with itself by rotating it about its center through the angle $4d/n$ in any direction. For instance (see Figure 230), rotating the hexagon 60° clockwise about O, we make the side AB go into BC, the side BC into CD, etc.

222. Problem. *To inscribe into a given circle:* (1) *a square*, (2) *a regular hexagon*, (c) *a regular triangle, and to express their sides through the radius of the circle.*

We will denote a_n the side of a regular n-gon inscribed into a circle of radius R.

(1) On Figure 231, two mutually perpendicular diameters AC and BD are drawn, and their endpoints are connected consecutively by chords. The resulting quadrilateral $ABCD$ is an inscribed square (because its angles are 90° each, and its diagonals are perpendicular). From the right triangle AOB we find, using the Pythagorean

theorem, that

$$a_4^2 = AB^2 = AO^2 + OB^2 = 2R^2, \quad \text{i.e. } a_4 = \sqrt{2}\,R = R \cdot 1.4142\ldots$$

(2) On Figure 232, a chord corresponding to a central angle of 60°, i.e. to the central angle of a regular hexagon, is shown. In the isosceles triangle AOB each of the angles A and B is $(180° - 60°)/2 = 60°$. Therefore the triangle is equiangular, and hence equilateral. Thus

$$AB = AO, \quad \text{i.e. } a_6 = R.$$

In particular we obtain a simple way of dividing a circle into 6 congruent parts by consecutively marking on it the endpoints of 6 chords, each 1 radius long.

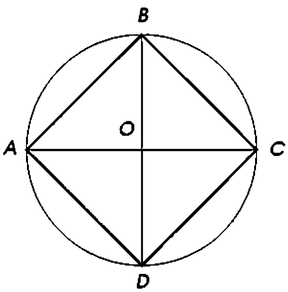

Figure 231

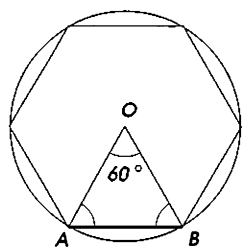

Figure 232

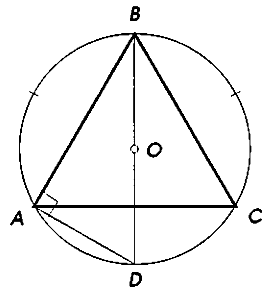

Figure 233

(3) To inscribe a regular triangle, divide a circle into 6 congruent parts (Figure 233), and then connect every other division point. The triangle ABC thus obtained is equilateral, and hence regular. Furthermore, draw the diameter BD and connect A and D to obtain a right triangle BAD. From the Pythagorean theorem, we find:

$$AB = \sqrt{BD^2 - AD^2} = \sqrt{(2R)^2 - R^2}, \text{ i.e. } a_3 = \sqrt{3}\,R = R{\cdot}1.7321\ldots$$

223. Problem. *To inscribe into a given circle a regular decagon and to express its side a_{10} through the radius R.*

Let us first prove the following important property of the regular 10-gon. Let AB (Figure 234) be a side of the regular 10-gon. Then the angle AOB contains 36°, and each of the angles A and B of the isosceles triangle AOB measures $(180° - 36°)/2 = 72°$. Bisect the angle A by the line AC. Then each of the angles formed at the vertex A contains 36°, and therefore $\triangle ACO$ is isosceles (as having two congruent angles), i.e. $AC = CO$, and $\triangle ABC$ is also isosceles (since $\angle B = 72°$, and $\angle ACB = 180° - 72° - 36° = 72°$), i.e. $AB = AC = CO$. By the property of the angle bisector (§184) we have the proportion: $AO : AB = CO : CB$. Replacing AO and AB with the congruent segments BO and CO, we obtain:

$$BO : CO = CO : CB.$$

In other words, the radius BO is divided at the point C in the extreme and mean ratio (§206), and CO is the greater part of it. Thus, *the side of a regular decagon inscribed into a circle is congruent to the greater part of the radius divided in the extreme and mean ratio.* In particular (see §206), the side a_{10} can be found from the quadratic equation:

$$x^2 + Rx - R^2 = 0, \text{ i.e. } a_{10} = x = \frac{\sqrt{5}-1}{2}R = R \cdot 0.6180\ldots$$

Now the construction problem is easily solved: divide a radius (e.g. OA) in the extreme and mean ratio as explained in §206, set the compass to the step congruent to the greater part of the radius, mark with this step 10 points around the circle one after another, and connect the consecutive division points by chords.

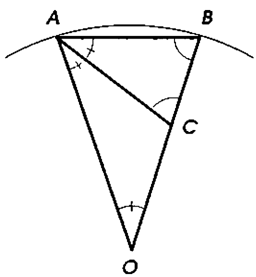

Figure 234

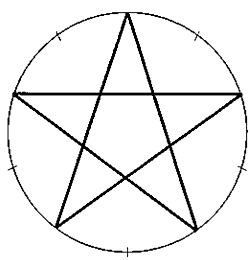

Figure 235

Remarks. (1) In order to inscribe into a given circle a regular pentagon, one divides the circle into 10 congruent parts and consecutively connects every other point by chords.

(2) The 5-point star can be constructed [1] similarly by dividing a circle into 10 congruent parts and connecting the division points skipping three at a time (Figure 235).

[1] In some countries, this problem is of national importance.— A.G.

(3) The equality

$$\frac{2}{5} - \frac{1}{3} = \frac{6}{15} - \frac{5}{15} = \frac{1}{15}$$

gives a simple way to inscribe a regular 15-gon, since we already know how to divide a circle into 5 and 3 congruent parts.

224. Problem. *To double the number of sides of an inscribed regular polygon.*

This is a concise formulation of two distinct problems: given an inscribed regular n-gon, (1) *to construct* a regular $2n$-gon inscribed into the same circle; (2) *to compute* the side of the $2n$-gon through the side of the n-gon and the radius of the circle.

(1) Let AB (Figure 236) be a side of a regular n-gon inscribed into a circle with the center O. Draw $OC \perp AB$ and connect A with C. The arc AB is bisected at the point C, and therefore the chord AC is a side of a regular $2n$-gon inscribed into the same circle.

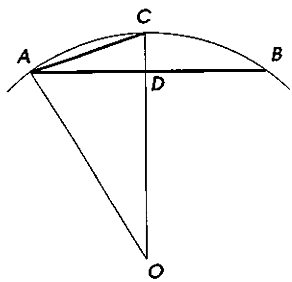

Figure 236

(2) In $\triangle AOC$, the angle O is acute (since the arc ACB is smaller than a semicircle, and hence the arc AC is smaller than a quarter-circle). Therefore the theorem of §190 applies:

$$a_{2n}^2 = AC^2 = OA^2 + OC^2 - 2OC \cdot OD = 2R^2 - 2R \cdot OD.$$

From the right triangle AOD, we find:

$$OD = \sqrt{OA^2 - AD^2} = \sqrt{R^2 - (a_n/2)^2} = \sqrt{R^2 - a_n^2/4}.$$

Thus

$$a_{2n}^2 = 2R^2 - 2R\sqrt{R^2 - \frac{a_n^2}{4}}.$$

The side a_{2n} is obtained from this **doubling formula** by extracting the square root.

`Example`. Let us compute the side of a regular 12-gon, taking for simplicity $R = 1$ (and therefore $a_6 = 1$). We have:

$$a_{12}^2 = 2 - 2\sqrt{1 - \frac{1}{4}} = 2 - 2\sqrt{\frac{3}{4}} = 2 - \sqrt{3}, \text{ i.e. } a_{12} = \sqrt{2 - \sqrt{3}}.$$

Since the sides of regular n-gons are proportional to their radii, then for the side of a regular 12-gon inscribed into a circle of an arbitrary radius R we obtain the formula:

$$a_{12} = R\sqrt{2 - \sqrt{3}} = R \cdot 0.517\ldots$$

225. Which regular polygons can be constructed by straightedge and compass? Applying the methods described in the previous problems, we can, using only straightedge and compass, divide a circle into a number of congruent parts (and hence construct the corresponding regular polygons) shown in the table:

3,	$3 \cdot 2$,	$3 \cdot 2 \cdot 2$,	...	generally	$3 \cdot 2^n$;
4,	$4 \cdot 2$,	$4 \cdot 2 \cdot 2$,	...	generally	2^n;
5,	$5 \cdot 2$,	$5 \cdot 2 \cdot 2$,	...	generally	$5 \cdot 2^n$;
15,	$15 \cdot 2$,	$15 \cdot 2 \cdot 2$,	...	generally	$3 \cdot 5 \cdot 2^n$.

A German mathematician C. F. Gauss (1777–1855) proved that using straightedge and compass, it is possible to divide a circle into only such a *prime* number of congruent parts, which is expressed by the formula $2^{2^n} + 1$. For instance, it is possible to divide a circle into 17 congruent parts, or 257 congruent parts, since 17 and 257 are prime numbers of the form $2^{2^n} + 1$ ($17 = 2^{2^2} + 1$; $257 = 2^{2^3} + 1$). A proof of Gauss' theorem requires methods which go beyond elementary mathematics.

It is also proved that using straightedge and compass one can divide a circle only into such a *composite* number of congruent parts which contains no other factors except: (1) prime factors of the form $2^{2^n} + 1$, in the first power; (2) the factor 2, in any power.

Whole numbers $F_n = 2^{2^n} + 1$ are called **Fermat numbers** after the remarkable French mathematician P. Fermat (1601–1665) who conjectured (erroneously) that all such numbers are prime. At present only the first five Fermat numbers are known to be prime:

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537.$$

EXERCISES

468. Find a formula for the side a_n of a regular n-gon inscribed into the circle of radius R for: (a) $n = 24$, (b) $n = 8$, (c) $n = 16$.

469. Find a formula for the sides of a regular triangle and regular hexagon circumscribed about a circle of a given radius.

470. Let AB, BC, and CD be three consecutive sides of a regular polygon with the center O. Prove that if the sides AB and CD are extended up to their intersection point E, then the quadrilateral $OAEC$ can be circumscribed by a circle.

471. Prove that: (a) every circumscribed equiangular polygon is regular; (b) every inscribed equilateral polygon is regular.

472. Give an example of: (a) a circumscribed equilateral quadrilateral which is not regular; (b) an inscribed equiangular quadrilateral which is not regular.

473. Prove that: (a) every circumscribed equilateral pentagon is regular; (b) every inscribed equiangular pentagon is regular.

*474.** For which n does there exist: (a) a circumscribed equilateral n-gon which is not regular; (b) an inscribed equiangular n-gon which is not regular?

475. Prove that two diagonals of a regular pentagon not issuing from the same vertex divide each other in the extreme and mean ratio.

*476.** Prove that if $ABCDEFG$ is a regular 7-gon, then $1/AB = 1/AC + 1/AD$.

*477.** Prove that the difference between the greatest and smallest diagonals of a regular 9-gon is congruent to its side.

478. Cut off the corners of a square in such a way that the resulting octagon is regular.

479. On a given side, construct a regular decagon.

480. Construct the angles: $18°$, $30°$, $72°$, $75°$, $3°$, $24°$.

481. Inscribe into a square a regular triangle so that one of its vertices is placed: (a) at a vertex of the square; (b) at the midpoint of one of its sides.

482. Into a given equilateral triangle, inscribe another equilateral triangle such that its side is perpendicular to a side of the given one.

483. Given a regular n-gon circumscribed about a given circle, construct a regular $2n$-gon circumscribed about the same circle.

*484.** Divide a given angle congruent to 1/7th of the full angle into: (a) three congruent parts; (b) five congruent parts.

2 Limits

226. Length of a curve. A segment of a straight line can be compared to another segment, taken for a unit, because straight lines can be superimposed onto each other. This is how we define which segments to consider congruent, which lengths equal, or unequal, what is the sum of segments, which segment is 2, 3, 4, ... times greater than the other, etc. Similarly, we can compare arcs of the same radius, because circles of the same radius can be superimposed. However no part of a circle (or another curve) can be superimposed onto a straight segment, which makes it impossible to decide this way which curvilinear segment should be assigned the same length as a given straight segment, and hence which curvilinear segment should be considered 2,3,4, ... times longer than the straight one. Thus we encounter the need to *define* what we mean by **circumference** as the *length* of a circle, when we compare it (or a part of it) to a straight segment.

For this, we need to introduce a concept of importance to all of mathematics, namely the concept of *limit.*

227. Limit of a sequence. In questions of algebra or geometry one often encounters a sequence of numbers following one another according to a certain pattern. For instance, the **natural series**:

$$1, \ 2, \ 3, \ 4, \ 5, \ \ldots,$$

arithmetic or **geometric progressions** extended indefinitely:

$$a, \ a+d, \ a+2d, \ a+3d, \ \ldots,$$

$$a, \ aq, \ aq^2, \ aq^3, \ \ldots$$

are examples of infinite sequences of numbers, or infinite **numerical sequences.**

For each such a sequence, one can point out a rule by which its terms are formed. Thus, in an arithmetical progression, each term differs from the previous one by the same number; in a geometric progression each two consecutive terms have the same ratio.

Many sequences are formed according to a more complex pattern. Thus, approximating $\sqrt{2}$ from below with the precision of up to: first 1/10, then 1/100, then 1/1000, and continuing such approximation indefinitely, we obtain the infinite numerical sequence:

$$1.4, \ 1.41, \ 1.414, \ 1.4142, \ \ldots$$

Although we do not give a simple rule that would determine each next term from the previous ones, it is still possible to define each term of the sequence. For example, to obtain the 4th term, one needs to represent $\sqrt{2}$ with the precision of 0.0001, to obtain the 5-th term, with the precision of 0.00001, and so on.

Suppose that the terms of an infinite numerical sequence

$$a_1, \; a_2, \; a_3, \; \ldots, \; a_n, \; \ldots$$

approach a certain number A as the index n increases indefinitely. This means the following: *there exists a certain number A such that however small a positive number q we pick, it is possible to find a term in the given sequence starting from which all terms of the sequence would differ from A by less than q in the absolute value.* We will briefly express this property by saying that the absolute value of the difference $a_n - A$ **tends** to 0) (or that the terms a_n tend to A) as n increases. In this case the number A is called the **limit** of a given numerical sequence.

For example, consider the sequence:

$$0.9, \quad 0.99, \quad 0.999, \quad \ldots,$$

where each term is obtained from the previous one by adding the digit 9 on the right. It is easy to see that the terms of this sequence tend to 1. Namely, the first term differs from 1 by 0.1, the second by 0.01, the third by 0.001, and continuing this sequence far enough, it is possible to find a term, starting from which all the following terms will differ from 1 by no more than a quantity, picked beforehand, as small as one wishes. Thus we can say that the infinite sequence in question has the limit 1.

Another example of a numerical sequence which has a limit is the sequence of consecutive approximations (say, from below) to the length of a segment (§151), computed with the precision of: first up to 1/10, then up to 1/100, then up to 1/1000, and so on. The limit of this sequence is the infinite decimal fraction representing the length of the segment. Indeed, the infinite decimal fraction is enclosed between two finite decimal approximations: one from above the other from below. As it was noted in §152, the difference between the approximations tends to 0 as the precision improves. Therefore the difference between the infinite fraction and the approximate values must also tend to 0 as the precision improves. Thus the infinite decimal fraction is the limit of each of the two sequences of its finite decimal approximations (one from above the other from below).

It is easy to see that not every infinite sequence has a limit; for instance, the natural series $1, 2, 3, 4, 5, \ldots$, obviously, does not have any limit since its terms increase indefinitely and therefore do not approach any number.

228. Theorem. ***Any infinite sequence has at most one limit.***

This theorem is easily proved by *reductio ad absurdum.* Indeed, suppose that we are given a sequence

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots,$$

which has two *distinct* limits A and B. Then, since A is a limit of the given sequence, the absolute value of the difference $a_n - A$ must tend to 0 as n increases. Since B is also a limit of the given sequence, the absolute value of the difference $a_n - B$ must also tend to 0 as n increases. Therefore the absolute value of the difference

$$(a_n - A) - (a_n - B)$$

for n sufficiently large must also tend to 0, i.e. become smaller than any number picked beforehand as small as one wishes. But this difference is equal to the difference $B - A$, and therefore it is a certain number different from 0. This number does not depend at all on the index n, and hence does not tend to 0 when n increases. Thus our assumption that there exist two limits of the numerical sequence leads to a contradiction.

229. The limit of an increasing sequence. Consider a sequence $a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$, such that each term of it is greater than the previous one (i.e. $a_{n+1} > a_n$), and at the same time all terms of which are smaller than a certain number M (i.e. $a_n < M$ for all values of the index n). In this case *the sequence has a limit.*

230. Proof. Let

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots, \qquad (*)$$

be a numerical sequence such that each term of it is greater than the previous one ($a_{n+1} > a_n$), and such that among terms of this sequence there is no one greater than a given number M, say, there is no term greater than 10. Take the number 9 and check if in the sequence $(*)$ there are terms greater than 9. Suppose that not. Then take the number 8 and check if in the sequence $(*)$ there are terms greater than 8. Suppose there are. Then write down the number 8, divide the interval from 8 to 9 into 10 equal parts, and test consecutively the numbers $8.1, 8.2, \ldots 8.9$, i.e. check if in the sequence $(*)$

there are terms greater than 8.1, and if yes, then decide the same question for 8.2, etc. Suppose that the sequence $(*)$ contains terms greater than 8.6, but contains no terms greater than 8.7. Then write down the number 8.6, divide the interval from 8.6 to 8.7 into 10 equal parts, and test consecutively the numbers $8.61, 8.62, \dots 8.69$. Suppose that the sequence $(*)$ contains terms greater than 8.64, but contains no terms greater than 8.65. Then write down the number 8.64, and proceed by dividing the interval from 8.64 to 8.65 into 10 equal parts, etc. Continuing this process indefinitely we arrive at an infinite decimal fraction: $8.64\dots$, i.e. at a certain real number. Denote this number by α, and denote its finite decimal approximations with n decimal places, from below and from above, by α_n and α'_n respectively. As it is known (§151),

$$\alpha_n \leq \alpha \leq \alpha'_n, \text{ and } \alpha'_n - \alpha_n = \frac{1}{10^n}.$$

From our construction of the real number α, it follows that the sequence $(*)$ contains no terms greater than α'_n but contains terms greater than α_n. Let a_k be one of such terms:

$$\alpha_n < a_k < \alpha'_n.$$

Since the sequence $(*)$ is increasing and contains no terms greater than α'_n, we find that all of the following terms of the sequence: a_{k+1}, a_{k+2}, ..., are also contained between α_n and a'_n, i.e. if $m > k$, then $\alpha_n < a_m < \alpha'_n$.

Since the real number α is also contained between α_n and α'_n, we conclude that for all $m \geq k$ the absolute value of the difference $a_m - \alpha$ does not exceed the difference $\alpha'_n - \alpha_n = 1/10^n$. Thus, for any value of n one can find the number k such that for all $m \geq k$ we have

$$|a_m - \alpha| < \frac{1}{10^n}.$$

Since the fraction $1/10^n$ tends to 0 as n indefinitely increases, it follows that the real number α is the limit of the sequence $(*)$.

EXERCISES

485. Express precisely what one means by saying that terms a_n of an infinite numerical sequence tend to a number A as n increases indefinitely.

486. Show that the sequence: $1, 1/2, 1/3, \dots, 1/n, \dots$ tends to 0.

487. Show that the sequence: $1, -1/2, 1/3, -1/4, \ldots, \pm 1/n, \ldots$ tends to 0.

488. Show that the natural series $1, 2, 3, \ldots, n, \ldots$ does not have a limit.

489. Show that the infinite sequence $1, -1, 1, -1, \ldots$ does not have a limit.

490. Formulate the rule describing which of two given infinite decimal fractions represents a greater number.

491. Which of the decimal fractions represents a greater number: (a) 0.099999 or 0.100000? (b) 0.099999... or 0.100000...?

*492.** Prove that if an infinite numerical sequence tends to a certain limit, then the sequence is **bounded**, i.e. all terms of the sequence lie in a certain segment of the number line.

493. Prove that a decreasing numerical sequence bounded below tends to a certain limit.

494. Show that an infinite geometric progression a, aq, aq^2, $\ldots$, tends to 0 provided that the absolute value of q is smaller than 1.

495. An ant crawled 1 m first, then 1/2 m more, then 1/4 m more, then 1/8 m more, etc. What is the total distance the ant crawled.

*496.** Compute the sum of an infinite geometric progression a, aq, aq^2, $\ldots$, provided that the absolute value of q is smaller than 1. Hint: First prove that the sum $a + aq + aq^2 + \cdots + aq^n$ of a finite geometric progression is equal to $a(1 - q^{n+1})/(1 - q)$.

3 Circumference and arc length

231. Two lemmas. The concept of limit gives us an opportunity to define precisely what we mean by the length of a circle. Let us first prove two lemmas.

Lemma 1. ***A convex broken line** ($ABCD$, Figure 237) **is shorter than any other broken line** ($AEFGD$) **enclosing the first one.***

The expressions "enclosing broken line" and "enclosed broken line" should be understood in the following sense. Let two broken lines (like those shown in Figure 237) have the same endpoints A and D and be situated in such a way that one broken line ($ABCD$) lies inside the polygon bounded by the other broken line together with the segment AD connecting the endpoints A and D. Then the outer broken line is referred to as *enclosing*, and the inner one as *enclosed*.

We intend to prove that the enclosed broken line $ABCD$, if it is convex, is shorter than any enclosing broken line (no matter convex or not), i.e. that

$$AB + BC + CD < AE + EF + FG + GD.$$

Extend the sides of the enclosed convex broken line as shown in Figure 237. Then, taking into account that a straight segment is shorter than any broken line connecting its endpoints, we can write the following inequalities:

$$\begin{aligned} AB + BH &< AE + EH; \\ BC + CK &< BH + HF + FG + GK; \\ CD &< CK + KD. \end{aligned}$$

Add all these inequalities and then subtract from both parts the auxiliary segments BH and CK. Then, replacing the sums $EH+HF$ and $GK+KD$ respectively with the segments EF and GD, we obtain the required inequality.

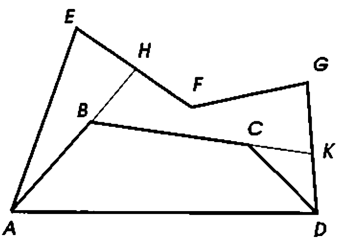

Figure 237

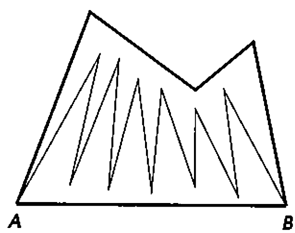

Figure 238

Remark. If the enclosed broken line were not convex (Figure 238), we would not be able to apply our argument. The enclosed line in this case can, indeed, turn out to be longer than the enclosing one.

Lemma 2. ***The perimeter of a convex polygon** ($ABCD$) **is smaller than the perimeter of any other polygon** ($MNPQRL$) **enclosing the first one*** (Figure 239).

It is required to prove that

$$AB + BC + CD + DA < LM + MN + NP + PQ + QR + RL.$$

Extending one of the sides AD of the enclosed convex polygon in both directions, and applying the previous lemma to the broken lines

$ABCD$ and $ATMNPQRSD$, connecting the points A and D, we obtain the inequality:

$$AB + BC < AT + TM + MN + NP + PQ + QR + RS + SD.$$

On the other hand, since the segment ST is shorter than the broken line SLT, we can write:

$$TA + AD + DS < TL + LS.$$

Add the two inequalities and subtract the auxiliary segments AT and DS from both parts. Then, replacing the sums $TL + TM$ and $LS + RS$ respectively with the segments LM and LR, we obtain the required inequality.

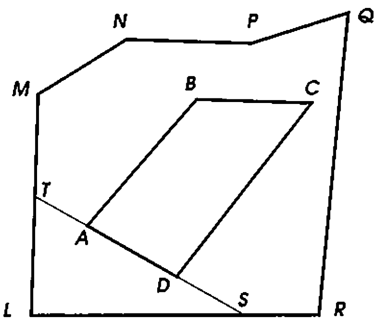

Figure 239

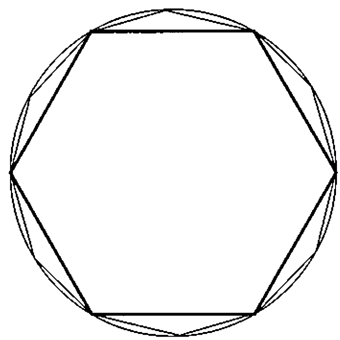

Figure 240

232. Definition of circumference. Inscribe into a given circle (Figure 240) a regular polygon, e.g. a hexagon, and mark on any line MN (Figure 241) the segment OP_1 congruent to the perimeter of this polygon.[2] Now double the number of sides of the inscribed polygon, i.e. replace the hexagon with the regular 12-gon, find its perimeter and mark it on the same line MN from the same point O. We obtain another segment OP_2, greater than OP_1 since each side of the hexagon is now replaced with a broken line (consisting of two sides of the 12-gon), which is longer than the straight line. Now double the number of sides of the 12-gon, i.e. take the regular 24-gon (*not* shown in Figure 240), find its perimeter, and mark it on the line MN from the same point O. We then obtain the segment OP_3, which will be greater than OP_2 (for the same reason that OP_2 is greater than OP_1).

[2]One may choose a unit of length and think of MN as a number line.

Imagine now that this process of doubling the number of sides of regular polygons and marking their perimeters on a line is continued indefinitely. Then we obtain an infinite sequence of perimeters OP_1, OP_2, OP_3, ..., which increases. However this increasing sequence is bounded, since perimeters of all inscribed convex polygons are smaller, according to Lemma 2, than the perimeter of any circumscribed polygon (as enclosing the inscribed ones). Therefore our increasing sequence of perimeters of inscribed regular polygons has a certain limit (§229). This limit (shown in Figure 241 as the segment OP) is taken for the **circumference**. Thus, *we define the circumference of a circle as the limit to which the perimeter of a regular polygon inscribed into the circle tends as the number of its vertices is doubled indefinitely.*

Figure 241

Remark. It is possible to prove (although we omit the proof) that this limit does not depend on the regular polygon the doubling procedure begins with. Moreover, it is possible to prove that even if the inscribed polygons are not regular, still their perimeters tend to the very same limit as the perimeters of the regular ones, if only their sides decrease indefinitely (and therefore the number of their sides indefinitely increases), no matter how this is achieved: by the doubling procedure we were using for regular polygons, or by any other rule. Thus, for any circle there exists a unique limit to which perimeters of inscribed polygons tend when all their sides decrease indefinitely, and this limit is taken for the circumference.

Similarly, the **arc length** of any arc AB (Figure 242) is defined as the limit to which the perimeter of a broken line, inscribed into the arc and connecting its endpoints A and B, tends when the sides of the broken line decrease indefinitely (e.g. by following the doubling procedure).

233. Properties of arc length. From the definition of arc length, we conclude:

(1) *Congruent arcs (and congruent circles) have equal arc length,* because the regular polygons inscribed into them, can be chosen congruent to each other.

(2) *The arc length of the sum of arcs is equal to the sum of their*

arc lengths.

Indeed, if s is the sum of two arcs s' and s'', then the broken line inscribed into the arc s can be chosen consisting of two broken lines: one inscribed into s', the other into s''. Then the limit to which the perimeter of such a broken line inscribed into s tends, as the sides of it indefinitely decrease, will be equal to the sum of the limits to which the perimeters of the broken lines inscribed into s' and s'' tend.

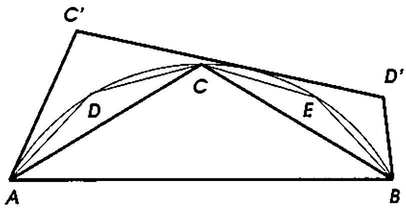

Figure 242

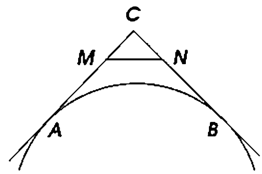

Figure 243

(3) *The arc length of any arc* (ACB, Figure 242) *is greater than the length of the chord AB connecting its endpoints, and more generally, than the perimeter of any convex broken line inscribed into the arc and connecting its endpoints.*

Indeed, by doubling the number of sides of the broken line and marking the perimeters on a number line we obtain an infinite sequence, which tends to the arc length, and is *increasing*. Therefore the arc length is greater than any of the terms of the sequence (in particular, than the first one of them, which is the length of the chord).

(4) *The arc length is smaller than the perimeter of any broken line circumscribed about the arc and connecting its endpoints.*

Indeed, the length L of the arc ACB (Figure 242) is the limit of the perimeters of regular broken lines ACB, $ADCEB$, etc. inscribed into the arc and obtained by the method of doubling. Each of these broken lines is convex and is enclosed by any circumscribed broken line $AC'D'B$ connecting the endpoints of the arc. Thus, by Lemma 1, the perimeters of the inscribed broken lines are smaller than the perimeter P of the circumscribed broken line, and therefore their limit L cannot exceed the perimeter P as well, i.e. $L \leq P$. In fact the same inequality will remain true if we replace the broken line $AC'D'B$ with a shorter broken line still enclosing the disk segment ACB. It is shown in Figure 243 how to construct such a shorter broken line by cutting the corner near one of the vertices (i.e. replacing the part

ACB between two consecutive tangency points by the shorter broken line $AMNB$). Therefore the arc length L is in fact strictly smaller than the perimeter P of the circumscribed broken line, i.e. $L < P$.

234. The number π. ***The ratio of the circumference to the diameter is the same number for all circles.***

Indeed, consider two circles: one of radius R, the other of radius r. Denote the circumference of the first circle C, and the second c. Inscribe into each of them a regular n-gon and denote P_n and p_n the respective perimeters. Due to similarity of regular polygons with the same number of sides, we have (see §220):

$$\frac{P_n}{2R} = \frac{p_n}{2r}. \qquad (*)$$

When the number n of sides doubles indefinitely, the perimeters P_n tend to the circumference C of the first circle, and the perimeters p_n to the circumference c of the second. Therefore the equality $(*)$ implies:

$$\frac{C}{2R} = \frac{c}{2r}.$$

This ratio of circumference to diameter, the same for all circles, is denoted by the Greek letter π. [3] Thus we can write the following formula for circumference:

$$C = 2R \cdot \pi, \text{ or } C = 2\pi R.$$

It is known that the number π is irrational and therefore cannot be expressed precisely by a fraction. However one can find rational approximations of π.

The following simple approximation of π, found by *Archimedes* in the 3rd century B.C., is sufficient for many practical purposes:

$$\pi \approx \frac{22}{7} = 3\frac{1}{7} = 3.142857142857\ldots$$

It is slightly greater than π, but by no more than 0.002. The Greek astronomer *Ptolemy* (in about 150 A.D.), and the author of *"Al-jabra," al-Khwarizmi of Baghdad* (in about 800 A.D.) found the approximation $\pi \approx 3.1416$ with the error of less than 0.0001. A Chinese

[3] The notation π, which became standard soon after it was adopted by L. Euler in 1737, comes from the first letter in the Greek word $\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$ meaning *circle*.

mathematician *Zu Chongzhi* (430–501) discovered that the following fraction:

$$\pi \approx \frac{355}{113} \approx 3.1415929\ldots,$$

approximates π from above with the remarkable precision of up to 0.0000005. [4]

235. A method of computation of π. To compute approximations to the number π, one can use the doubling formula we derived in §224. For simplicity, take the radius R of a regular n-gon equal to 1. Let a_n denote the side of the n-gon, and $q_n = na_n/2$ its *semi*-perimeter, which tends therefore to π as the number of sides is doubled indefinitely. According to the doubling formula,

$$a_{2n}^2 = 2 - 2\sqrt{1 - \frac{a_n^2}{4}}.$$

We can begin the computation with $a_6 = 1$ (i.e. $q_6 = 3$). Then the doubling formula yields (see §224):

$$a_{12}^2 = 2 - \sqrt{3} = 0.26794919\ldots$$

Using the doubling formula we then consecutively compute:

$$a_{24}^2 = 2 - 2\sqrt{1 - \frac{a_{12}^2}{4}}, \quad a_{48}^2 = 2 - 2\sqrt{1 - \frac{a_{24}^2}{4}}, \quad \text{and so on.}$$

Suppose that we stop the doubling at the 96-gon, and take its semi-perimeter $q_{96}/2 = 48a_{96}$ for an approximate value of π. Performing the computation, we find:

$$\pi \approx q_{96} = 3.1410319\ldots$$

In order to judge the precision of this approximation, let us also compute the semi-perimeter Q_{96} of the 96-gon *circumscribed* about the circle of the unit radius. Applying the formula for the side of circumscribed regular polygons found in §220, and setting $R = 1$ we get:

$$b_{96} = \frac{a_{96}}{\sqrt{1 - a_{96}^2/4}}, \text{ i.e. } Q_{96} = 48b_{96} = \frac{q_{96}}{\sqrt{1 - a_{96}^2/4}}.$$

[4] In 1883, an Englishman W. Shanks published his computation of π with 707 decimal places. It held the record until 1945, when the first 2000 places were found using computers, and it turned out that Shanks had made a mistake which ruined his results starting with the 528th decimal place.

Substituting numerical values of a_{96} and q_{96} we find:

$$Q_{96} = 3.1427146\ldots$$

A semicircle is greater than the semi-perimeter of the inscribed regular 96-gon, but smaller than the semi-perimeter of the circumscribed regular 96-gon: $q_{96} < \pi < Q_{96}$. Thus we can conclude that $3.141 < \pi < 3.143$. In particular, we find the decimal approximation to π from below true to two decimal places:

$$\pi \approx 3.14.$$

More precise approximations of π can be found by using the same method of doubling for computing q_{192} and Q_{192}, q_{384} and Q_{384}, and so on. For instance, to obtain the approximation from below

$$\pi \approx 3.141592\ldots$$

true to 6 decimal places, i.e. with the precision of up to 0.000001, it suffices to compute semi-perimeters of regular inscribed and circumscribed polygons with 6144 sides (which are obtained from hexagons by 10 doublings).

236. Radian. In some problems, the number inverse to π occurs:

$$\frac{1}{\pi} = 0.3183098\ldots$$

Problem. *Determine the number of degrees in an arc whose arc length is equal to the radius.*

The formula $2\pi R$ for circumference of a circle of radius R means that the arc length of one degree is equal to $2\pi R/360 = \pi R/180$. Therefore an arc of n degrees has the arc length

$$s = \frac{\pi R n}{180}.$$

When the arc length is equal to the radius, i.e $s = R$, we obtain the equation $1 = \pi n/180$, from which we find:

$$n^\circ = \frac{1}{\pi}180^\circ \approx 180^\circ \cdot 0.3183098 \approx 57.295764^\circ \approx 57^\circ 17' 45''.$$

An arc whose arc length is equal to the radius is called a **radian**. Radians are often used (instead of circular and angular degrees) as units for measuring arcs and corresponding central angles. For instance, the full angle contains 360° or 2π radians.

EXERCISES

497. Compute the length of the arcs of the unit radius subtended by the chords: (a) $\sqrt{2}$ units long; (b) $\sqrt{3}$ units long.

498. Compute the radian measure of the angles containing: 60°, 45°, 12°.

499. Express in radians the sum of the interior angles of an n-gon.

500. Express in radians the exterior and interior angles of a regular n-gon.

501. How many degrees are contained in the angle whose radian measure is: π, $\pi/2$, $\pi/6$, $3\pi/4$, $\pi/5$, $\pi/9$?

502. Compute the values of the trigonometric functions $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, and $\cot\alpha$ for the angles $\alpha = \pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$, π radians.

*503.** Prove that $\sin\alpha < \alpha < \tan\alpha$ for $0 < \alpha < \pi/2$, where α denotes the radian measure of the angle.

504. Prove that in two circles, the ratio of central angles corresponding to two arcs of the same arc length is equal to the inverse ratio of the radii.

505. Two tangent lines at the endpoints of a given arc containing 120° are drawn, and a circle is inscribed into the figure bounded by these tangent lines and the arc. Prove that the circumference of this circle is equal to the arc length of the given arc.

506. In a circle, the arc subtended by a chord of length a is congruent to twice the arc subtended by a chord of length b. Compute the radius of the circle.

507. Prove that the side a_n of a regular n-gon tends to 0 as the number of sides increases indefinitely.

508. On the diameter of a given semicircle, inside the disk segment bounded by the diameter and the semicircle, two congruent semicircles tangent to each other are constructed. Into the part of the plane bounded by the three semicircles, a disk is inscribed. Prove that the ratio of the diameter of this disk to the diameter of the constructed semicircles is equal to 2 : 3.

509. How small will the error be if instead of semi-circumference we take the sum of the side of an inscribed equilateral triangle and the side of an inscribed square?

510. Estimate the length of the Earth's equator, taking the Earth's radius to be 6400 *km*.

511. Estimate the length of 1° of the Earth's equator.

512. A round rope, which is 1 m longer than the Earth's equator, is stretched around the equator at a constant height above the Earth's surface. Can a cat squeeze itself between the rope and the Earth's surface?

513.* Suppose now that the same rope is stretched around the equator and pulled up at one point as high as possible above the Earth's surface. Can an elephant pass under the rope?

Chapter 5

AREAS

1 Areas of polygons

237. The concept of area. We all have some idea about the quantity called *area*, from everyday life. For example, the harvest a farmer expects to collect from a piece of land depends not so much on the shape of the piece, but only on the size of land surface that the farmer cultivates. Likewise, to determine the amount of paint needed to paint a surface, it suffices to know the overall size of the surface rather than the exact shape of it.

We will establish here more precisely the concept of area of geometric figures, and develop methods for its computation.

238. Main assumptions about areas. We will assume that the area of a geometric figure is a quantity, expressed by *positive* numbers, and is well-defined for *every polygon.* We further assume that the areas of figures possess the following properties:

(1) *Congruent figures have equal areas.* Figures of equal area are sometimes called **equivalent**. Thus, according to this property of areas, *congruent figures are equivalent.* The converse can be false: equivalent figures are not always congruent.

(2) *If a given figure is partitioned into several parts* (M, N, P, Figure 244), *then the number expressing the area of the whole figure is equal to the sum of the numbers expressing the areas of the parts.* This property of areas is called **additivity**. It implies, that *the area of any polygon is greater than the area of any other polygon enclosed by it.* Indeed, the difference between the areas of the enclosing and enclosed polygons is positive since it represents the area of a figure (namely of the remaining part of the enclosing polygon, which can

always be partitioned into several polygons).

(3) *The square, whose side is a unit of length, is taken for the unit of area,* i.e. the number expressing the area of such a square is set to 1. Of course, which squares have unit areas depends on the unit of length. When the unit of length is taken to be, say, 1 meter (centimeter, foot, inch, etc.), the unit square of the corresponding size is said to have the area of 1 **square** meter (respectively square centimeter, square foot, square inch, etc.), which is abbreviated as 1 m^2 (respectively cm^2, ft^2, in^2, etc.)

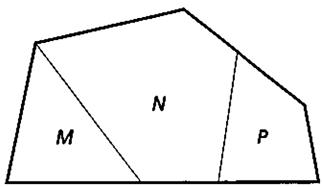

Figure 244

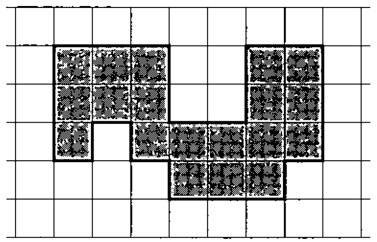

Figure 245

239. Mensuration of areas. Area of some simple figures can be measured by counting the number of times the unit square fits into the figure. For example, let the figure in question be drawn on *grid paper* (Figure 245) made of unit squares, and suppose that the boundary of the given figure is a closed broken line whose sides coincide with the edges of the grid. Then the whole number of unit squares lying inside the figure gives the exact measure of the area.

In general, measuring areas is done not by direct counting of unit squares or their parts fitting into the measured figure, but indirectly, by means of measuring certain linear sizes of the figure, as it will be explained soon.

240. Base and altitude. Let us agree to call one of the sides of a triangle or parallelogram the **base** of these figures, and a perpendicular dropped to this side from the vertex of the triangle, or from any point of the opposite side of the parallelogram, the **altitude**.

In a rectangle, the side perpendicular to the base can be taken for the altitude.

In a trapezoid, both parallel sides are called bases, and a common perpendicular between them, an altitude.

The base and the altitude of a rectangle are called its **dimensions**.

241. Theorem. ***The area of a rectangle is the product of its dimensions.***

This brief formulation should be understood in the following way: the number expressing the area of a rectangle in certain square units is equal to the product of the numbers expressing the length of the base and the altitude of the rectangle in the corresponding linear units.

In the proof of this theorem, three cases can occur:

(i) The lengths of the base and the altitude (measured by the same unit) are expressed by *whole* numbers.

Let a given rectangle (Figure 246) have the base equal to b linear units, and the altitude to h such units. Divide the base and the altitude into respectively b and h congruent parts, and draw through the division points two series of lines parallel respectively to the altitude and the base. Mutual intersections of these lines partition the rectangle into quadrilaterals. In fact each of these quadrilaterals (e.g. **K**) is congruent to the unit square. (Indeed, since the sides of **K** are parallel to the sides of the rectangle, then all angles of **K** are right; and the lengths of the sides of **K** are equal to the distances between the parallel lines, i.e. to the same linear unit.) Thus the rectangle is partitioned into squares of unit area each, and it remains to find the number of these squares. Obviously, the series of lines parallel to the base divides the rectangle into as many rectangular strips as there are linear units in the altitude, i.e. into h congruent strips. Likewise, the series of lines parallel to the altitude divides each of the strips into as many unit squares as there are linear units in the base, i.e. into b such squares. Therefore the total number of squares is $b \times h$. Thus

$$\text{the area of a rectangle} = bh,$$

i.e. it is equal to the product of the base and the altitude.

(ii) The length of the base and the altitude (measured by the same unit) are expressed by *fractions.*

Suppose, for example, that in a given rectangle:

$$\begin{aligned} \text{base } &= 3\frac{1}{2} = \frac{7}{2} \text{ linear units,} \\ \text{altitude } &= 4\frac{3}{5} = \frac{23}{5} \text{ of the same linear units.} \end{aligned}$$

Bringing the fractions to a common denominator, we obtain:

$$\text{base } = \frac{35}{10}; \quad \text{altitude } = \frac{46}{10}.$$

Let us take the $\frac{1}{10}$th part of the linear unit for a new unit of length. Then we can say that the base contains 35 such units, and the altitude 46. Thus, by the result of case (i), the area of the rectangle is equal to 35×46 square units corresponding to the new unit of length. But this square unit is equivalent to the $\frac{1}{100}$th part of the square unit corresponding to the original unit of length. Therefore the area of the rectangle, expressed in the original square units, is equal to

$$\frac{35 \times 46}{100} = \frac{35}{10} \times \frac{46}{10} = \left(3\frac{1}{2}\right) \times \left(4\frac{3}{5}\right).$$

(iii) The base and the altitude (or only one of these dimensions) are incommensurable with the unit of length, and therefore are expressed by *irrational* numbers.

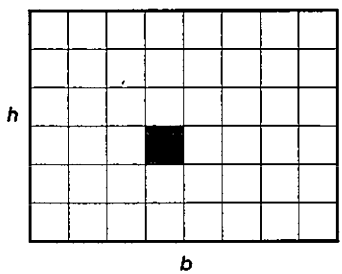

Figure 246

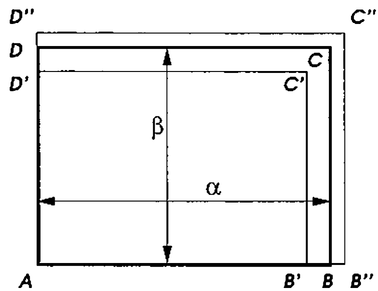

Figure 247

For all practical purposes it suffices to use approximate values of the area computed with any desired precision. It is possible however to show that in this case too, the precise value of the area of the rectangle is equal to the product of its dimensions.

Indeed, let the lengths of the base AB and the altitude AD of a rectangle $ABCD$ (Figure 247) be expressed by real numbers α and β. Let us find the approximate values of α and β with the precision of up to $1/n$. For this, mark on the base AB the $\frac{1}{n}$th part of the linear unit as many times as possible. Suppose, that marking m such parts, we obtain a segment $AB' < AB$ (or $AB' = AB$), and marking $m+1$ such parts, we obtain a segment $AB'' > AB$. Then the fractions $\frac{m}{n}$ and $\frac{m+1}{n}$ will be the approximations of α respectively from below and from above, with the required precision. Furthermore, suppose that by marking on AD the $\frac{1}{n}$th part of the unit p and $p+1$ times, we obtain the segments respectively $AD' < AD$ (or $AD' = AD$) and

$AD'' > AD$, and thus find the approximations $\frac{p}{n} \leq \beta < \frac{p+1}{n}$ to the length β of the altitude. Construct two auxiliary rectangles $AB'C'D'$ and $AB''C''D''$. The dimensions of each of them are expressed by rational numbers. Therefore, by case (ii): the area of $AB'C'D'$ is equal to $\frac{m}{n} \times \frac{p}{n}$, and the area of $AB''C''D''$ is equal to $\frac{m+1}{n} \times \frac{p+1}{n}$. Since $ABCD$ encloses $AB'C'D'$ and is enclosed by $AB''C''D''$, we have:

$$\text{area of } AB'C'D' < \text{area of } ABCD < \text{area of } AB''C''D'',$$

$$\text{i.e.} \quad \frac{m}{n} \times \frac{p}{n} < \text{area of } ABCD < \frac{m+1}{n} \times \frac{p+1}{n}.$$

This inequality holds true for any value of n, i.e. with whatever precision we choose to approximate α and β. Let us first take $n = 10$, then $n = 100$, then $n = 1000$, etc. We will obtain the fractions $\frac{m}{n}$ and $\frac{p}{n}$ which provide better and better decimal approximations of the numbers α and β from below, and the fractions $\frac{m+1}{n}$ and $\frac{p+1}{n}$ which provide better and better approximations of the numbers α and β from above. It is not hard to see that their products become better and better approximations, from below and from above, of *the same* infinite decimal fraction.[1] The latter decimal fraction represents the real number called the **product** of the real numbers α and β. Thus, we conclude that the area of $ABCD$ is equal to $\alpha\beta$.

242. Theorem. ***The area of a parallelogram*** ($ABCD$, Figure 248) ***is equal to the product of the base and the altitude.***

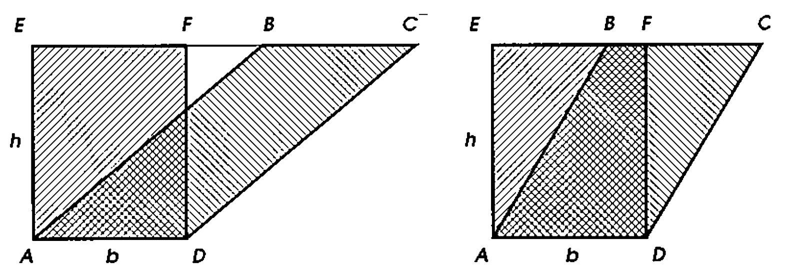

Figure 248

[1]Indeed, the difference

$$\frac{m+1}{n}\frac{p+1}{n} - \frac{m}{n}\frac{p}{n} = \frac{mp+m+p+1-mp}{n^2} = \frac{1}{n}\left(\frac{m}{n} + \frac{p+1}{n}\right) = \frac{AB' + AD''}{n}$$

tends to zero as n increases indefinitely.

On the base AD, construct the rectangle $AEFD$, whose side EF extends the side BC of the parallelogram, and prove (in both cases shown in Figure 248), that

$$\text{area of } ABCD = \text{ area of } AEFD.$$

Namely, combining the parallelogram with the triangle AEB, and the rectangle with the triangle DFC, we obtain the same trapezoid $AECD$. The triangles AEB and DFC are congruent (by the SAS-test, since $AE = DF$, $AB = DC$, and $\angle EAB = \angle FDC$), they are equivalent, and therefore the parallelogram and the rectangle have to be equivalent as well. But the area of $AEFD$ is equal to bh, and hence the area of $ABCD$ is equal to bh as well, where b can be considered as the base, and h as the altitude of the parallelogram.

243. Theorem. ***The area of a triangle*** (ABC, Figure 249) ***is equal to half the product of the base and the altitude.***

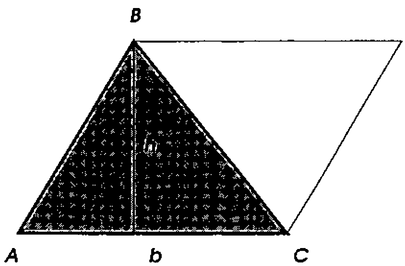

Figure 249

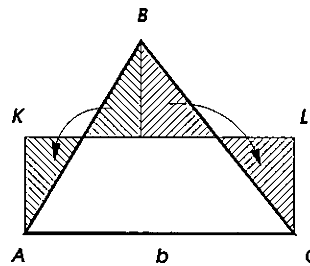

Figure 250

Drawing $BD \| AC$ and $CD \| AB$, we obtain the parallelogram $ABDC$ whose area, by the previous theorem, is equal to the product of the base and the altitude. But the parallelogram consists of two congruent triangles, one of which is $\triangle ABC$. Thus

$$\text{area of } \triangle ABC = \frac{bh}{2}.$$

Remark. Figure 250 shows how to rearrange parts of a triangle ABC to form the rectangle $AKLC$ with the same base b as the triangle, and the altitude $h/2$ congruent to a half of the altitude of the triangle.

244. Corollaries. (1) *Triangles with congruent bases and congruent altitudes are equivalent.*

For example, if we will move the vertex B of the triangle ABC (Figure 251) along the line parallel to the base AC, leaving the base unchanged, then the area of the triangle will remain constant.

(2) *The area of a right triangle is equal to half the product of its legs,* because one of the legs can be taken for the base, the other for the altitude.

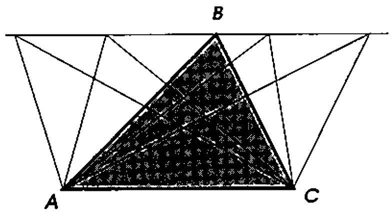

Figure 251

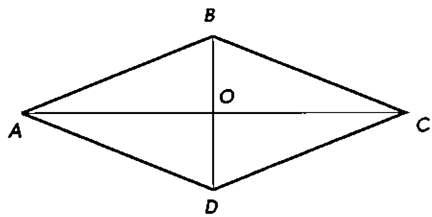

Figure 252

(3) *The area of a rhombus is equal to half the product of its diagonals.* Indeed, if $ABCD$ (Figure 252) is a rhombus, then its diagonals are perpendicular. Therefore

$$\text{area of } \triangle ABC = \frac{1}{2} AC \cdot OB, \quad \text{area of } \triangle ADC = \frac{1}{2} AC \cdot OD,$$

$$\text{i.e.} \quad \text{area of } ABCD = \frac{1}{2} AC \cdot (OB + OD) = \frac{1}{2} AC \cdot BD.$$

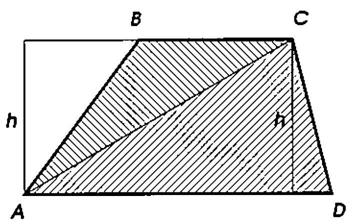

Figure 253

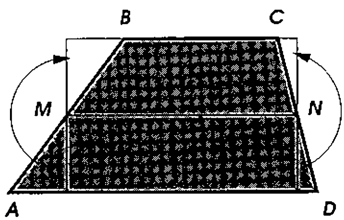

Figure 254

245. Theorem. ***The area of a trapezoid is equal to the product of the altitude and the semi-sum of the bases.***

Drawing in the trapezoid $ABCD$ (Figure 253) the diagonal AC, we can consider the area of the trapezoid as the sum of areas of the triangles ACD and BAC. Therefore

$$\text{area of } ABCD = \frac{1}{2} AD \cdot h + \frac{1}{2} BC \cdot h = \frac{1}{2}(AD + BC) \cdot h.$$

246. Corollary. If MN (Figure 254) is the midline of the trapezoid $ABCD$, then (as it is known from §97) it is congruent to the semi-sum of the bases. Therefore

$$\text{area of } ABCD = MN \cdot h,$$

i.e. *the area of a trapezoid is equal to the product of the midline with the altitude.*

This can also be seen directly from Figure 254.

247. Remark. In order to find the area of an arbitrary polygon, one can partition it into triangles, compute the area of each triangle, and add the results.

EXERCISES

Prove theorems:

514. In a parallelogram, the distances from any point of a diagonal to two adjacent sides are inversely proportional to these sides.

515. A convex quadrilateral each of whose diagonals divides it into two equivalent triangles, is a parallelogram.

516. In a trapezoid partitioned into four triangles by the diagonals, the triangles adjacent to the lateral sides are equivalent.

517. The area of a trapezoid is equal to the product of one of the lateral sides and the perpendicular, dropped to this side from the midpoint of the other lateral side.

518. A triangle with the altitudes 12, 15, and 20 *cm* is right.

519. The parallelogram obtained from intersection of the lines connecting each vertex of a given parallelogram with the midpoint of the next side is equivalent to 1/5th of the given parallelogram.

*520.** If the medians of one triangle are taken for the sides of another, then the area of the latter triangle is equal to 3/4 of the area of the former one.

*521.** In a quadrilateral $ABCD$, through the midpoint of the diagonal BD, the line parallel to the diagonal AC is drawn. Suppose that this line intersects the side AD at a point E. Prove that the line CE bisects the area of the quadrilateral.

Computation problems

522. In a square with the side a, midpoints of adjacent sides are connected to each other and to the opposite vertex. Compute the area of the triangle thus formed.

523. Two equilateral triangles are inscribed into a circle of radius R in such a way that each of the sides is divided by the intersections with the sides of the other triangle into 3 congruent parts. Compute the area of the common part of these triangles.

524. Compute the area of a right triangle, if the bisector of an acute angle divides the opposite leg into segments of lengths 4 and 5.

525. Compute the area of a trapezoid with angles $60°$ and $90°$, given: (a) both bases, (b) one base and the lateral side perpendicular to the bases, (c) one base and the other lateral side.

526. Given the bases and the altitude of a trapezoid, compute the altitude of the triangle formed by the extensions of the lateral sides up to the point of their intersection.

*527.** Compute the area of an isosceles trapezoid with perpendicular diagonals, if the midline is given.

*528.** Compute the ratio of the area of a triangle to the area of another triangle whose sides are congruent to the medians of the former triangle.

529. Into a triangle of unit area, another triangle, formed by the midlines of the first triangle, is inscribed. Into the second triangle, a third triangle, formed by the midlines of the second one, is inscribed. Into the third triangle, a fourth one is inscribed in the same fashion, and so on indefinitely. Find the limit of the sum of the areas of these triangles.
Hint: First compute the sum of the areas after finitely many steps.

Construction problems

530. Through a vertex of a triangle, draw two lines which divide the area in a given proportion $m : n : p$.

531. Bisect the area of a triangle by a line passing through a given point on its side.

532. Find a point inside a triangle such that the lines connecting the point with the vertices divide the area of the triangle (a) into three equal parts; (b) in a given proportion $m : n : p$.

533. Divide a parallelogram into three equivalent parts by lines drawn from one of its vertices.

534. Divide the area of a parallelogram in a given proportion $m : n$ by a line passing through a given point.
Hint: Divide a midline of the parallelogram in the given proportion, and connect the division point with the given one.

2 Several formulas for areas of triangles

248. Theorem. ***The area of any circumscribed polygon is equal to the product of the semi-perimeter of the polygon and the radius.***

Connecting the center O (Figure 255) with all vertices of the circumscribed polygon, we partition it into triangles, in which sides of the polygon can be taken for the bases, and radii for the altitudes. If r denotes the radius, then

$$\text{area of } \triangle AOB = \frac{1}{2}AB \cdot r, \quad \text{area of } \triangle BOC = \frac{1}{2}BC \cdot r, \quad \text{etc.}$$

i.e. $$\text{area of } ABCDE = \frac{1}{2}(AB + BC + CD + DE + EF) \cdot r = qr,$$

where the letter q denotes the semi-perimeter of the polygon.

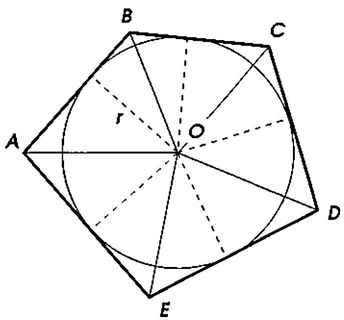

Figure 255

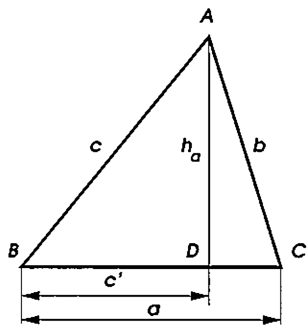

Figure 256

Corollaries. (1) *The area of a regular polygon is equal to the product of the semi-perimeter and the apothem,* because any regular polygon can be considered as circumscribed about a circle the radius of which is the apothem of the polygon.

(2) *The area S of any triangle is equal to the product of its semi-perimeter q and the radius r of the inscribed circle:*

$$S = qr.$$

249. Problem. *To compute the area S of a triangle, given the lengths a, b, and c of its sides.*

Let h_a denote the altitude of $\triangle ABC$ (Figure 256) dropped to its side a. Then

$$S = \frac{1}{2}ah_a.$$

In order to compute the altitude h_a, we use the relation (§190):

$$b^2 = a^2 + c^2 - 2ac',$$

and determine from it c':

$$c' = \frac{a^2 + c^2 - b^2}{2a}.$$

From the right triangle ADB, we find:

$$h_a = \sqrt{c^2 - \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2} = \frac{1}{2a}\sqrt{4a^2c^2 - (a^2 + c^2 - b^2)^2}.$$

Transform the expression under the square root sign:

$$\begin{aligned}(2ac)^2 - (a^2 + c^2 - b^2)^2 &= (2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)\\ &= [(a^2 + c^2 + 2ac) - b^2][b^2 - (a^2 + c^2 - 2ac)]\\ &= [(a + c)^2 - b^2][b^2 - (a - c)^2]\\ &= (a + c + b)(a + c - b)(b + a - c)(b - a + c).\end{aligned}$$

Therefore [2]

$$S = \frac{1}{2}ah_a = \frac{1}{4}\sqrt{(a + b + c)(a + b - c)(a + c - b)(b + c - a)}.$$

Let $q = (a+b+c)/2$ denote the semi-perimeter of the triangle. Then

$$a + c - b = (a + b + c) - 2b = 2q - 2b = 2(q - b),$$

and similarly

$$a + b - c = 2(q - c), \quad b + c - a = 2(q - a).$$

Thus

$$S = \frac{1}{4}\sqrt{2q \cdot 2(q - a) \cdot 2(q - b) \cdot 2(q - c)},$$

i.e.

$$S = \sqrt{q(q - a)(q - b)(q - c)}.$$

The last expression is known as **Heron's formula** after *Heron of Alexandria* who lived in the 1st century A.D.

[2] Since any side of a triangle is smaller than the sum of the other two sides, the factors under the square root sign are positive.

Example. The area of an *equilateral* triangle with the side a is given by the formula

$$S = \sqrt{\frac{3a}{2} \cdot \frac{a}{2} \cdot \frac{a}{2} \cdot \frac{a}{2}} = \frac{\sqrt{3}}{4} a^2.$$

250. The law of sines.[3]

Theorem. ***The area of a triangle is equal to half the product of any two of its sides and the sine of the angle between them.***

Indeed, the altitude h_a (Figure 257) of $\triangle ABC$ can be expressed as $h_a = b \sin C$, and therefore the area S of the triangle is given by the formula

$$S = \frac{1}{2} ab \sin C.$$

The following corollary is called the **law of sines.**

Corollary. *Sides of a triangle are proportional to the sines of the angles opposite to them:*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Indeed, from the theorem, we compute $\sin C = 2S/ab$, and find the ratio

$$\frac{c}{\sin C} = \frac{abc}{2S}.$$

It follows that the ratio is the same for all three sides of the triangle.

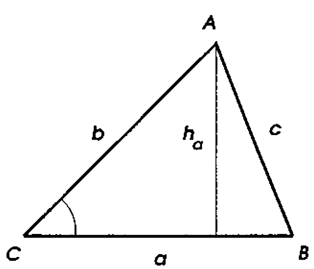

Figure 257

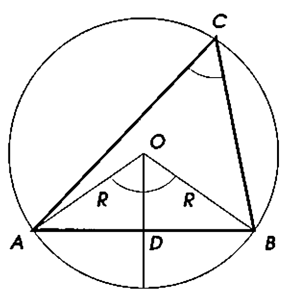

Figure 258

The following theorem provides another proof of the law of sines.

[3] See also Exercises in Section 7 of Chapter 3.

Theorem. ***Any side of a triangle is equal to the product of the sine of the opposite angle and the diameter of the circumscribed circle.***

Let O (Figure 258) be the center of the circle circumscribed about $\triangle ABC$, and OD the perpendicular bisector of the side AB. The central angles AOD and BOD are congruent to each other and to $\angle C$ (because they are all measured by a half of the arc ADB). Since $AO = OB = R$ (where by R we denote the radius of the circle), then $AD = DB = R\sin C$, i.e.

$$c = AB = 2R\sin C.$$

Corollaries. (1) *The ratio of any side of a triangle to the sine of the opposite angle, is equal to the diameter of the circumscribed circle:*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

(2) Comparing two expressions for the ratio $c/\sin C$, we obtain a simple *formula expressing the area S of a triangle through its sides a, b, c and the radius R of the circumscribed circle:*

$$S = \frac{abc}{4R}.$$

EXERCISES

Prove theorems:

535. The area of any quadrilateral is equal to half the product of its diagonals and the sine of the angle between them.

536. If the areas of two triangles, adjacent to the bases of a trapezoid and formed by the intersection of the diagonals, are equal to a^2 and b^2 respectively, then the area of the whole trapezoid is equal to $(a+b)^2$.

537. The area S of a triangle with the sides a, b, c and the semiperimeter q can be expressed as

$$S = (q-a)r_a = (q-b)r_b = (q-c)r_c,$$

where r_a, r_b, and r_c are radii of the exscribed circles tangent to the sides a, b, and c respectively.

538. Prove that the radii r_a, r_b, r_c, and r of the three exscribed and one inscribed circle of a triangle satisfy: $1/r_a + 1/r_b + 1/r_c = 1/r$.

539. The medians of a given triangle divide it into six triangles, out of which two adjacent to one side of the given triangle turned out to have congruent inscribed circles. Prove that the given triangle is isosceles.

540. A line dividing a given triangle into two figures which have equal areas and congruent perimeters, passes through the incenter.

541. In a convex equilateral polygon, the sum of distances from an interior point to the sides or their extensions does not depend on the point.

542.* In an equiangular polygon, the sum of distances from an interior point to the sides or their extensions is a quantity independent of the position of the point in the polygon.

543.* The sum of the squares of the distances from a point on a circle to the vertices of an inscribed equilateral triangle is a quantity independent of the position of the point on the circle.

Computation problems

544. Compute the area of a regular hexagon with the side a.

545. Compute the area of a regular 12-gon of radius R.

546. A disk inscribed into an isosceles trapezoid touches a lateral side at a point dividing it into segments m and n. Compute the area of the trapezoid.

547. Express the radius of the circumscribed circle of a triangle in terms of two sides of the triangle and the altitude dropped to the third side.

548. Three circles of radii 6, 7, and 8 cm are pairwise tangent to each other. Compute the area of the triangle formed by the three lines of centers.

549. Express the common chord of two intersecting circles in terms of their radii and the line of centers.

550. Express the radius of the inscribed circle of a triangle, and each of its exscribed circles, through the sides of the triangle.

551. Express the radius of the circumscribed circle of a triangle through the sides.

552. If the lengths a, b, c of the sides of a triangle form an arithmetic sequence, then $ac = 6Rr$, where R and r are the radii of the circumscribed and inscribed circles respectively.

3 Areas of similar figures

251. Theorem. ***Areas of similar triangles or polygons are proportional to the squares of homologous sides.***

(i) If ABC and $A'B'C'$ (Figure 259) are two similar triangles, then their areas are equal to respectively $ah/2$ and $a'h'/2$, where a and a' are lengths of the homologous sides BC and $B'C'$, and h and h' are the homologous altitudes AD and $A'D'$.

The altitudes are proportional to the homologous sides: $h : h' = a : a'$ (since from similarity of the right triangles ADB and $A'D'B'$, we have $h : h' = c : c' = a : a'$). Therefore

$$\frac{\text{area of } ABC}{\text{area of } A'B'C'} = \frac{ah}{a'h'} = \frac{a}{a'} \cdot \frac{h}{h'} = \frac{a}{a'} \cdot \frac{a}{a'} = \frac{a^2}{(a')^2}.$$

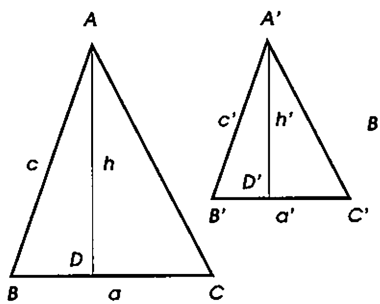

Figure 259

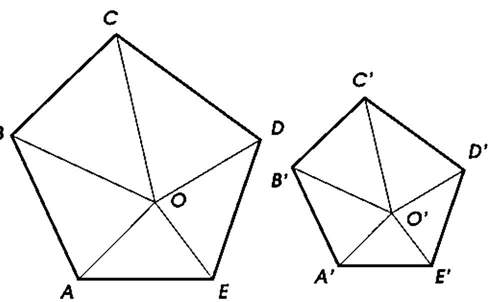

Figure 260

(ii) If $ABCDE$ and $A'B'C'D'E'$ (Figure 260) are two similar polygons, then it is possible, as we have seen in §168, to partition them into respectively similar triangles positioned in the same way. Let these triangles be: AOB and $A'O'B'$, BOC and $B'O'C'$, etc. According to the result of part (i), we have the following proportions:

$$\frac{\text{area of } AOB}{\text{area of } A'O'B'} = \left(\frac{AB}{A'B'}\right); \quad \frac{\text{area of } BOC}{\text{area of } B'O'C'} = \left(\frac{BC}{B'C'}\right); \text{ etc.}$$

But from the similarity of the polygons, we have:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \ldots, \text{ and hence } \left(\frac{AB}{A'B'}\right)^2 = \left(\frac{BC}{B'C'}\right)^2 = \ldots$$

Therefore

$$\frac{\text{area of } AOB}{\text{area of } A'O'B'} = \frac{\text{area of } BOC}{\text{area of } B'O'C'} = \dots$$

From properties of proportions (see Remark in §169), we conclude:

$$\frac{\text{area of } AOB \ + \text{ area of } BOC \ + \ \dots}{\text{area of } A'O'B' \ + \text{ area of } B'O'C' \ + \ \dots} = \frac{\text{area of } AOB}{\text{area of } A'O'B'},$$

$$\text{i.e.} \qquad \frac{\text{area of } ABCDE}{\text{area of } A'B'C'D'E'} = \frac{AB^2}{(A'B')^2}.$$

Corollary. *Areas of regular polygons with the same number of sides are proportional to the squares of their sides, or squares of their radii, or squares of their apothems.*

252. Problem. *To divide a given triangle into m equivalent parts by lines parallel to one of its sides.*

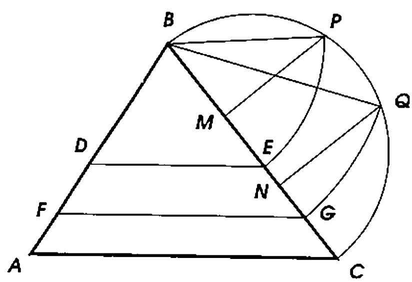

Figure 261

Suppose, for example, that it is required to divide $\triangle ABC$ (Figure 261) into three equivalent parts by segments parallel to AC. Suppose that the required segments are DE and FG. The triangles DBE, FBG, and ABC are similar. Therefore

$$\frac{\text{area of } DBE}{\text{area of } ABC} = \frac{BE^2}{BC^2} \text{ and } \frac{\text{area of } FBG}{\text{area of } ABC} = \frac{BG^2}{BC^2}.$$

But

$$\frac{\text{area of } DBE}{\text{area of } ABC} = \frac{1}{3} \text{ and } \frac{\text{area of } FBG}{\text{area of } ABC} = \frac{2}{3}.$$

Therefore

$$\frac{BE^2}{BC^2} = \frac{1}{3} \text{ and } \frac{BG^2}{BC^2} = \frac{2}{3}.$$

From this, we find:

$$BE = \sqrt{\frac{1}{3}BC \cdot BC} \text{ and } BG = \sqrt{\frac{2}{3}BC \cdot BC},$$

i.e. BE is the geometric mean between BC and $\frac{1}{3}BC$, and BG is the geometric mean between BC and $\frac{2}{3}BC$. Therefore the required construction can be done as follows. Divide BC into 3 congruent parts at the points M and N. Describe the semicircle on BC as the diameter. From the points M and N, erect the perpendiculars MP and NQ. The chords BP and BQ will be the geometric means needed: the first one between the diameter BC and its third part BM, the second one between BC and BN, i.e. between BC and $\frac{2}{3}BC$. It remains to mark these chords on BC starting from the point B to obtain the required points E and G.

One can similarly divide the triangle into any number of equivalent parts.

EXERCISES

Computation problems

553. A line parallel to the base of a triangle divides its area in the proportion $4 : 5$ counting from the vertex. In what proportion does it divide the lateral sides?

554. Each median of a triangle is divided in the proportion $3 : 1$ counting from the vertex. Compute the ratio of the area of the triangle with the vertices at the division points, to the area of the original triangle.

555.* Among rectangles of a fixed area, find the one with the minimal perimeter.

Construction problems

556. Divide a parallelogram into three equivalent parts by lines parallel to one of the diagonals.

557. Divide the area of a triangle in the extreme and mean ratio by a line parallel to the base.
Hint: Apply the algebraic method.

558.* Divide a triangle into three equivalent parts by lines perpendicular to the base.

559. Bisect the area of a trapezoid by a line parallel to the bases.

560. On a given base, construct a rectangle equivalent to a given one.

561. Construct a square equivalent to 2/3 of the given one.

562. Transform a given square into an equivalent rectangle with a given sum (or difference) of two adjacent sides.

563. Given two triangles, construct a third one, similar to the first, and equivalent to the second.

564. Transform a given triangle into an equivalent equilateral one. Hint: Apply the algebraic method.

565. Into a given disk, inscribe a rectangle of a given area a^2.

566. Into a given triangle, inscribe a rectangle of a given area S.

4 Areas of disks and sectors

253. Lemma. ***Under unlimited doubling of the number of sides of an inscribed regular polygon, its side decreases indefinitely.***

Let n be the number of sides of an inscribed regular polygon, and p its perimeter. Then the length of one of its sides is expressed by the ratio p/n. Under unlimited doubling of the number of sides of the polygon, the denominator n of this ratio will increase indefinitely, and the numerator p will also increase, though not indefinitely (since the perimeter of any convex inscribed polygon remains smaller than the perimeter P of any fixed circumscribed polygon). A ratio, whose numerator remains bounded, and denominator increases indefinitely, tends to zero. Therefore the side of the inscribed regular polygon indefinitely decreases as n indefinitely increases.

254. Corollary. Let AB (Figure 262) be a side of an inscribed regular polygon, OA the radius, and OC the apothem. From $\triangle AOC$ we find:

$$OA - OC < AC, \quad \text{i.e.} \quad OA - OC < \frac{1}{2}AB.$$

Since the side of the regular polygon, as we have just proved, decreases indefinitely when the number of sides is doubled an unlimited number of times, then the same is true for the difference $OA - OC$. Therefore, *under unlimited doubling of the number of sides of the inscribed regular polygon, the length of the apothem tends to the radius.*

255. The area of a disk. Into a disk, whose radius we denote R, inscribe any regular polygon. Let the area of this polygon be S, semi-perimeter q, and apothem r. We have seen in §248 that

$$S = qr.$$

Imagine now that the number of sides of this polygon is doubled indefinitely. Then the semi-perimeter q and the apothem r (and hence the area S) will increase. The semi-perimeter will tend to the limit $C/2$ equal to the semi-circumference of the circle, and the apothem r will tend to the limit equal to the radius R. It follows that the area of the polygon will tend to the limit equal to $\frac{1}{2}C \cdot R$.

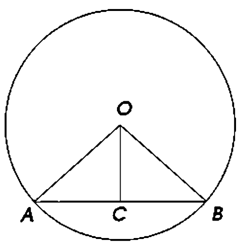

Figure 262

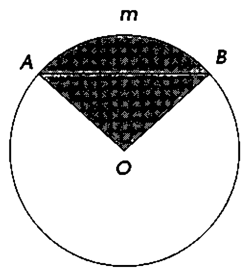

Figure 263

Definition. *The limit, to which the area of a regular polygon inscribed into a given disk tends as the number of sides of the polygon is doubled indefinitely, is taken for the area of the disk.*

Let us denote by A the area of the disk. We conclude therefore, that

$$A = \frac{1}{2}C \cdot R,$$

i.e. ***the area of a disk is equal to the product of the semi-circumference and the radius.***

Since $C = 2\pi R$, then

$$A = \frac{1}{2}2\pi R \cdot R = \pi R^2,$$

i.e. ***the area of a disk of radius R is equal to the square of the radius multiplied by the ratio of the circumference to the diameter.***

Corollary. *Areas of disks are proportional to the squares of their radii or diameters.*

Indeed, if A and A' denote areas of two disks of radii R and R' respectively, then $A = \pi R^2$ and $A' = \pi(R')^2$. Therefore

$$\frac{A}{A'} = \frac{\pi R^2}{\pi(R')^2} = \frac{R^2}{(R')^2} = \frac{4R^2}{4(R')^2} = \frac{(2R)^2}{(2R')^2}.$$

256. Area of a sector. ***The area of a sector is equal to half the product of its arc length and the radius.***

Let the arc AmB (Figure 263) of a sector AOB contain n°. Obviously, the area of the sector, whose arc contains 1°, is equal to $1/360$th part of the area of the disk, i.e. it is equal to $\pi R^2/360$. Therefore the area S of the sector, whose arc contains n°, is equal to

$$S = \frac{\pi R^2 n}{360} = \frac{1}{2}\frac{\pi R n}{180} \cdot R.$$

The fraction $\pi Rn/180$ expresses the arc length of the arc AmB (§236). If s denotes the arc length, then

$$S = \frac{1}{2} sR.$$

Remark. In order to find the area of the disk segment, bounded by an arc AmB (Figure 263) and the chord AB, it suffices to compute separately the area of the sector AOB and $\triangle AOB$, and then to subtract the latter from the former one.

257. Problem. *To compute the area of the disk whose circumference is equal to* 2 *cm.*

First, we find the radius R from the equation

$$2\pi R = 2\ cm, \text{ i.e.} R = \frac{1}{\pi} = 0.3183\dots\ cm.$$

Then we find the area of the disk:

$$A = \pi R^2 = \pi \cdot \left(\frac{1}{\pi}\right)^2 = \frac{1}{\pi} = 0.3183\dots\ cm^2.$$

258. Problem. *To construct the square equivalent to a given disk.*

This is the famous problem of **squaring the circle**. In fact it cannot be solved by means of straightedge and compass. If a square with the side x is equivalent to the disk of radius R, then

$$x^2 = \pi R^2, \text{ i.e. } x = \sqrt{\pi}R.$$

Let us assume for simplicity that $R = 1$. If the square with the side $x = \sqrt{\pi}$ could be constructed, then, according to the results of §213, the number $\sqrt{\pi}$ would have been expressible through integers by means of arithmetic operations and square roots. However, in 1882 a German mathematician *Ferdinand Lindemann* proved that π is **transcendental**. By definition, this means that it is not a solution of any polynomial equation with integer coefficients. In particular, this implies that it cannot be obtained from integers by arithmetic operations and extractions of roots.

For the same reason, the problem of constructing a segment whose length would be equal to the circumference of a given circle, also cannot be solved by means of straightedge and compass.

EXERCISES

567. In a disk with the center O, a chord AB is drawn, and another disk is constructed on the line OA as a diameter. Prove that the areas of two disk segments cut off by the chord AB from the two disks have the ratio $4:1$.

568. Construct a disk equivalent to a given **ring** (i.e. the figure bounded by two concentric circles).

569. Divide a disk into 2, 3, etc. equivalent parts by concentric circles.

570. Compute the area of the disk segment cut off by a side a of an inscribed into the disk: (a) equilateral triangle, (b) square, (c) regular hexagon.

571. Compute the ratio of the area of a sector intercepting a $60°$ arc to the area of the disk inscribed into this sector.

572. Compute the area of the figure bounded by three pairwise tangent congruent circles of radius R and situated in the exterior of the circles.

573. The common chord of two disks subtends the arcs of $60°$ and $120°$ respectively. Compute the ratio of the areas of these disks.

574. Compute the area of a ring if the chord of the outer boundary circle tangent to the inner boundary circle has length a.

575. Prove that if the diameter of a semicircle is divided into two arbitrary segments, and another semicircle is described on each of the segments as the diameter, then the figure bounded by the three semicircles is equivalent to the disk whose diameter is congruent to the perpendicular to the diameter of the original semicircle erected at the division point.

5 The Pythagorean theorem revisited

259. Theorem. ***The areas of squares constructed on the legs of a right triangle add up to the area of the square constructed on its hypotenuse.***

This proposition is yet another form of the Pythagorean theorem, which we proved in §188: *the square of the number measuring the length of the hypotenuse is equal to the sum of the squares of the numbers measuring the legs.* Indeed, the square of the number measuring the length of a segment is the number measuring the area of the square constructed on this segment.

There are many other ways to prove the Pythagorean theorem.

Euclid's proof. Let ABC (Figure 264) be a right triangle, and $BDEA$, $AFGC$, and $BCKH$ squares constructed on its legs and the hypotenuse. It is required to prove that the areas of the first two squares add up to the area of the third one.

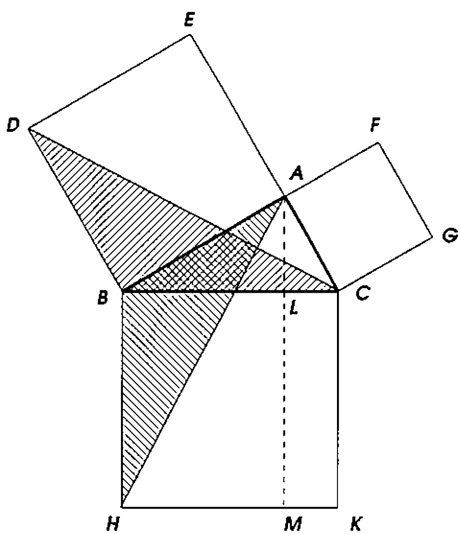

Figure 264

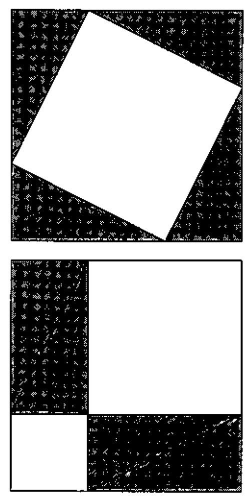

Figure 265

Draw $AM \perp BC$. Then the square $BCKH$ is divided into two rectangles. Let us prove that the rectangle $BLMH$ is equivalent to the square $BDEA$, and the rectangle $LCKM$ is equivalent to the square $AFGC$. For this, consider two triangles shaded in Figure 264. These triangles are congruent, since $\triangle ABH$ is obtained from $\triangle DBC$ by clockwise rotation about the point B through the angle of 90°. Indeed, rotating this way the segment BD, which is a side of the square $BDEA$, we obtain another side BA of this square, and rotating the segment BC, which is a side of the square $BCKH$, we obtain BH. Thus ΔABH and ΔDBC are equivalent. On the other hand, ΔDBC has the base DB, and the altitude congruent to BA (since $AC \| DB$). Therefore ΔDBC is equivalent to a half of the square $BDEA$. Likewise, ΔABH has the base BH, and the altitude congruent to BL (since $AL \| BH$). Therefore ΔABH is equivalent to a half of the rectangle $BLMH$. Thus the rectangle $BLMH$ is equivalent to the square $BDEA$. Similarly, connecting G with B, and A with K, and considering ΔGCB and ΔACK, we prove that the rectangle $LCKM$ is equivalent to the square $AFGC$. This implies that the square $BCKH$ is equivalent to the sum of the squares $BDEA$ and $AFGC$.

`A tiling proof`, shown in Figure 265, is based on tiling the square, whose side is congruent to the sum of the legs of a given right triangle, by the square constructed on the hypotenuse and by four copies of the given triangle, and then re-tiling it by the squares constructed on the legs and by the same four triangles.

One more proof, based on similarity, will be explained shortly.

260. Generalized Pythagorean theorem. The following generalization of the Pythagorean theorem is found in the 6th book of Euclid's "Elements."

`Theorem.` ***If three similar polygons*** (**P**, **Q**, and **R**, Figure 266) ***are constructed on the sides of a right triangle, then the polygon constructed on the hypotenuse is equivalent to the sum of the polygons constructed on the legs.***

In the special case when the polygons are squares, this proposition turns into the Pythagorean theorem as stated in §259. Due to the theorem of §251, the generalization follows from this special case. Indeed, the areas of similar polygons are proportional to the squares of homologous sides, and therefore

$$\frac{\text{area of } \mathbf{P}}{a^2} = \frac{\text{area of } \mathbf{Q}}{b^2} = \frac{\text{area of } \mathbf{R}}{c^2}.$$

Then, by properties of proportions,

$$\frac{\text{area of } \mathbf{P} + \text{area of } \mathbf{Q}}{a^2 + b^2} = \frac{\text{area of } \mathbf{R}}{c^2}.$$

Since $a^2 + b^2 = c^2$, it follows that

$$\text{area of } \mathbf{P} \ + \ \text{area of } \mathbf{Q} \ = \ \text{area of } \mathbf{R}.$$

Moreover, the same reasoning applies to similar figures more general than polygons. However, Euclid gives another proof of the generalized Pythagorean theorem, which does not rely on this special case. Let us explain such a proof here. In particular, we will obtain one more proof of the Pythagorean theorem itself.

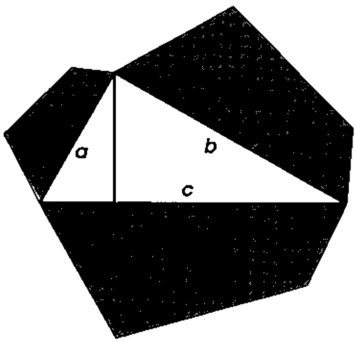

Figure 266

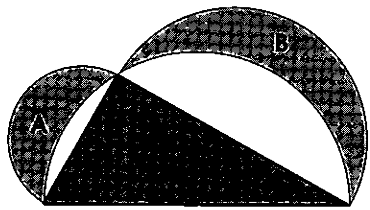

Figure 267

First, let us notice that to prove the generalized Pythagorean theorem, it suffices to prove it for polygons of *one shape only.* Indeed, suppose that the areas of two polygons **R** and **R**$'$ of different shapes constructed on some segment (e.g. the hypotenuse) have a certain ratio k. Then the areas of polygons similar to them (e.g. **P** and **P**$'$, or **Q** and **Q**$'$) and constructed on another segment which is, say, m times shorter, will be m^2 times smaller for both shapes. Therefore they will have the same ratio k. Thus, if the areas of **P**$'$, **Q**$'$ and **R**$'$ satisfy the property that the first two add up to the third one, then the same holds true for the areas of **P**, **Q** and **R** which are k times greater.

Now the idea is to take polygons similar not to a square, but to the right triangle itself, and to construct them not outside the triangle but inside it.[4]

Namely, drop the altitude of the right triangle to its hypotenuse. The altitude *divides* the triangle into two triangles similar to it. Together with the original triangle, we thus have three similar right triangles constructed on the sides of it, and such that two of the areas add up to the third one.

Corollary. *If outside of a right triangle* (Figure 267) *two semicircles are described on its legs, and another semicircle is described*

[4]The collage on the cover of this book illustrates this idea.

on the hypotenuse so that it contains the triangle, then the geometric figure bounded by the semicircles is equivalent to the triangle:

$$\text{area of } \mathbf{A} \ + \ \text{area of } \mathbf{B} \ = \ \text{area of } \mathbf{C}.$$

Indeed, after adding to both sides of this equality the areas (unshaded in Figure 267) of the disk segments bounded by the greatest of the semicircles and by the legs of the triangle, it is required to prove that the areas of the half-disks constructed on the legs add up to the area of the half-disk constructed on the hypotenuse. This equality follows from the generalized Pythagorean theorem.

Remark. The figures **A** and **B** are known as **Hippocrates' lunes** after a Greek mathematician *Hippocrates of Chios* who studied them in the 5th century B.C. in connection with the problem of squaring the circle. When the triangle is isosceles, then the lunes are congruent and each is equivalent to a half of the triangle.

EXERCISES

Miscellaneous problems

576. The altitude dropped to the hypotenuse divides a given right triangle into smaller triangles whose radii of the inscribed circles are 6 and 8 *cm*. Compute the radius of the inscribed circle of the given triangle.

577. Compute the sides of a right triangle given the radii of its circumscribed and inscribed circle.

578. Compute the area of a right triangle if the foot of the altitude dropped to the hypotenuse of length c divides it in the extreme and mean ratio.

579. Compute the area of the quadrilateral bounded by the four bisectors of the angles of a rectangle with the sides a and b *cm*.

*580.** Cut a given rectangle into four right triangles so that they can be reassembled into two smaller rectangles similar to the given one.

581. The diagonals divide a quadrilateral into four triangles of which three have the areas 10, 20, and 30 cm^2, and the area of the fourth one is greater. Compute the area of the quadrilateral.

582. A circle of the radius congruent to the altitude of a given isosceles triangle is rolling along the base. Show that the arc length cut out on the circle by the lateral sides of the triangle remains constant.

583. A circle is divided into four arbitrary arcs, and the midpoints of the arcs are connected pairwise by straight segments. Prove that two of the segments are perpendicular.

584. Compute the length of a common tangent of two circles of radii r and $2r$ which intersect at the right angle.

585. Prove that in a triangle, the altitudes h_a, h_b, h_c, and the radius of the inscribed circle satisfy the relation: $1/h_a + 1/h_b + 1/h_c = 1/r$.

586. Prove that in a right triangle, the sum of the diameters of the inscribed and circumscribed circles is congruent to the sum of the legs.

*587.** Prove that in a scalene triangle, the sum of the diameters of the inscribed and circumscribed circle is congruent to the sum of the segments of the altitudes from the orthocenter to the vertices.

*588.** Find the geometric locus of all points with a fixed difference of the distances from the sides of a given angle.

*589.** A side of a square is the hypotenuse of a right triangle situated in the exterior of the square. Prove that the bisector of the right angle of the triangle passes through the center of the square, and compute the distance between the center and the vertex of the right angle of the triangle, given the sum of its legs.

*590.** From each of the two given points of a given line, both tangents to a given circle are drawn, and in the two angles thus formed, congruent circles are inscribed. Prove that their line of centers is parallel to the given line.

*591.** Three congruent circles intersect at one point. Prove that the three lines, each passing through the center of one of the circles and the second intersection point of the other two circles, are concurrent.

*592.** Given a triangle ABC, find the geometric locus of points M such that the triangles ABM and ACM are equivalent.

*593.** On a given circle, find two points, A and B, symmetric about a given diameter CD and such that a given point E on the diameter is the orthocenter of the triangle ABC.

*594.** Find the geometric locus of the points of intersection of two chords AC and BD of a given circle, where AB is a fixed chord of this circle, and CD is any chord of a fixed length.

*595.** Construct a triangle, given its altitude, bisector and median drawn from the same vertex.

*596.** Construct a triangle, given its circumcenter, incenter, and the intersection point of the extension of one of the bisectors with the circumscribed circle.

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